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THE STABILITY OF MOTION OF SATELLITES WITH FLEXIBLE APPENDAGES

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16. Abstract

The mathematical formulation associated with the problem of stability of motion of a satellite consisting of a main rigid body and three (or less) pairs of flexible rods is presented. The rods are capable of flexure in two orthogonal directions. Whereas the rotational motion of the body is described by generalized coordinates depending on time alone, the elastic displacements of the rods depend both on spatial position and time. Assuming no external torques, there exist motion integrals in the form of momentum integrals. These integrals can be regarded as constraint equations relating the system velocities, and used to reduce the number of variables describing the motion.

The stability analysis has been carried out by means of an extension of the Liapunov direct method. Since the elastic vibrations result in energy dissipation, it is shown that the equilibrium position is asymptotically stable if the Hamiltonian is positive definite and unstable if it can take negative values in the neighborhood of the equilibrium. Determining the sign definiteness of the Hamiltonian is complicated by the fact that it contains spatial derivatives of the elastic displacements. Two methods are presented to cope with this problem. The first, the standard modal analysis in conjunction with series truncation, develops criteria in terms of infinite series truncation, develops criteria in terms of infinite series associated with the natural modes and frequencies of the elastic rods. The second, the method of integral coordinates, yields closed-form stability criteria involving the system parameters, such as the body moments of inertia, the length and mass distribution of the rods, the lowest natural frequencies of the rods, and the satellite spin velocity. The advantage of the method of integral coordinates is illustrated by the relative ease with which closed-form stability criteria are developed and by the amount of information which can be extracted from their ready physical interpretation.

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Introduction

The rotational motion of a torque-free rigid body is known to be stable if the rotation takes place about an axis corresponding to the maximum or minimum moment of inertia, but the motion is unstable if the rotation takes place about an axis of intermediate principal moment of inertia (see, for example, the text by Meirovitch[1], Sec. 6.7). In a large number of investigations concerned with the attitude stability of spinning spacecraft, the spacecraft is envisioned as a rotating, torque-free rigid body. It is assumed that the spacecraft dimensions, although finite, are small compared with the distance to the center of force. This mathematical model permits the assumption that the attitude motion has no effect upon the orbital motion, thus reducing the complexity of the problem by regarding the orbital motion as known. But in general spacecraft are not entirely rigid and the question remains as to what extent the rigid-body idealization can be justified. A number of investigations concerned with the dynamics of satellites containing elastic parts have indeed been conducted. In the sequel some of these studies are reviewed as a way of introducing the present problem.

In an attempt to explain the tumbling motion of the Explorer I satellite, Thomson and Reiter[2] and Meirovitch[3] have investigated the effect of energy dissipation resulting from the vibration of certain elastic parts of the satellite. On the basis of energy considerations, these investigations concluded that, for spin stabilization, spinning motion must be imparted to the satellite about the axis of maximum moment of inertia. Later works by Auelmann[4], Pringle[5], and Likins[6] established the usefulness of the Liapunov direct method for the investigation of the attitude stability of satellites, at least for the case of rigid satellites. Subsequently, Pringle[7] used the Liapunov direct method to investigate the stability of a body with connected moving parts. The formulation of Reference 7, however, is based entirely on ordinary differential equations and is suitable for investigating discrete systems but not distributed ones.

More pertinent to the present subject is the work by Meirovitch and Nelson[8] who investigated the stability of motion of a satellite containing elastic parts by means of an infinitesimal analysis. Reference 8 represents one of the first attempts to treat rigorously distributed elastic members. The displacement of the elastic members is represented as a series of normal modes multiplying time-dependent generalized coordinates and the effect of truncating the series on the system stability is explored. Also related to the present problem is the one of a satellite with elastically connected moving parts investigated by Nelson and Meirovitch[9] via the Liapunov direct method. In this work the distributed elastic members are simulated by means of discrete masses. The dynamics of a spacecraft consisting of two rigid bodies joined by an elastic structure has been investigated by Robe and Kane.[10]

Ignoring gravitational terms, an infinitesimal analysis is carried out for small motions about the simple-spin equilibrium position. The dynamics of satellites containing elastic parts has been further studied by Likins and Wirsching.[11] This latter work considers a discrete system and employs the normal modes to represent elastic displacements.

The Liapunov direct method has been widely used to analyze the stability of discrete systems. In recent years, however, work has been done on extending the Liapunov method to distributed-parameter systems. In this regard we single out the works by Wang[12,13] and by Parks[14] who applied the method to analyze the stability of partial differential equations associated with elastic and aeroelastic systems. From Refs. 12-14 it can be concluded that one of the major problems in applying the Liapunov direct method to continuous systems is that of constructing a suitable testing function. (Actually the same statement can be made in connection with discrete systems.)

The motion of spinning bodies containing distributed elastic members is described by sets of both ordinary and partial differential equations. We refer to such sets of differential equations as "hybrid". In Reference 12 Wang presents a simple example of a hybrid system. In a first attempt to apply Liapunov's direct method to hybrid systems from the area of satellite dynamics, Meirovitch[15,16] studied the stability of a spinning rigid body with elastic appendages. Several new concepts were introduced in Ref. 15, such as the use of some of Rayleigh's quotient properties to eliminate the dependence of the testing functional on the spatial derivatives, as well as the concept of a testing density function. Reference 16 extends the theory to torque-free hybrid systems.

This present study extends the work of Refs. 15 and 16 to the case of hybrid systems in which testing density functions cannot be readily defined. The mathematical model consists of a torque-free spinning rigid body with three pairs of rigidly-attached flexible rods. First the Hamiltonian equations of motion, with appropriate boundary conditions, are derived. The stability analysis follows the pattern of Ref. 15, in which it is shown that under certain circumstances the system Hamiltonian H is a suitable Liapunov functional. Through the use of certain properties of Rayleigh's quotient, it is possible to define a new functional κ, such that $H \geq \kappa$, and to prove that if κ is positive definite in the neighborhood of the origin, then the trivial solution is asymptotically stable. In contrast to the method of Ref. 15, in this case it is not possible to define an appropriate testing density function. Two approaches are presented here to circumvent this difficulty. The first, modal analysis in conjunction with series truncation, leads to stability criteria in terms of infinite series. The second method involves defining new time-dependent coordinates in terms of certain integrals appearing in the system Hamiltonian. Using these integral coordinates and Schwarz's inequality for functions it is possible to discretize the testing functional κ and test its sign properties by using Sylvester's criterion. This method yields closed-form stability criteria lending themselves to ready physical interpretation.

General Problem Formulation

Let us consider a body of total mass m moving relative to an inertial space XYZ, as shown in Figure 1. The entire body or parts of the body are capable of small elastic deformations from a reference equilibrium position coinciding with the undeformed state of the body. Next we define two sets of body axes, the set xyz with the origin at point 0 and coinciding with the principal axes of the body in the undeformed state, and the set $\xi\eta\zeta$ which is parallel to xyz but has the origin at the center of mass c of the deformed body. We note that $\xi\eta\zeta$ is not a principal set of axes. The set xyz serves as a suitable reference frame for measuring elastic deformations whereas the set $\xi\eta\zeta$ is more convenient for expressing the overall motion. The position of a typical point in the undeformed body relative to axes xyz is denoted by the vector* $\underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$ and the elastic displacement of an element of mass dm, originally coincident with that point, by the vector $\underset{\sim}{u} = u(x,y,z,t)\underset{\sim}{i} + v(x,y,z,t)\underset{\sim}{j} + w(x,y,z,t)\underset{\sim}{k}$, where $\underset{\sim}{i},\underset{\sim}{j},\underset{\sim}{k}$ are unit vectors along

---

* Vector quantities are denoted by wavy lines under the symbols.

axes x,y,z (or axes ξ,η,ζ), respectively. The radius vector from point $\underline{0}$ to c is given by $\underline{r}_c = \frac{1}{m}\int_m (\underline{r}+\underline{u})\,dm = \frac{1}{m}\int_m \underline{u}\,dm$, where we note that $\int_m \underline{r}\,dm$ is zero by virtue of the fact that $\underline{0}$ is the center of mass of the undeformed body. All integrations denoted by $\int_m$ --- dm are carried over the domain occupied by the body in undeformed state, which is designated as the reference state.

From Figure 1 we conclude that the position of the mass element dm relative to the inertial space is $\underline{R}_d = \underline{R}_c + \underline{r} + \underline{u}_c$, where $\underline{u}_c = \underline{u} - \underline{r}_c = u_c\underline{i} + v_c\underline{j} + w_c\underline{k}$ represents the displacement vector measured with respect to axes $\xi\eta\zeta$ and $\underline{R}_c$ is the position of the origin of these axes relative to the inertial space. Assuming that axes xyz, hence also axes $\xi\eta\zeta$, rotate with angular velocity $\underline{\omega} = \omega_\xi\underline{i} + \omega_\eta\underline{j} + \omega_\zeta\underline{k}$ relative to the inertial space, and denoting by $\dot{\underline{u}}_c' = \dot{u}_c\underline{i} + \dot{v}_c\underline{j} + \dot{w}_c\underline{k}$ the velocity of dm relative to $\xi\eta\zeta$ due to the elastic effect, we have $\dot{\underline{r}} + \dot{\underline{u}}_c = \dot{\underline{u}}_c' + \underline{\omega}\times(\underline{r} + \underline{u}_c)$.

In view of the above definitions, the kinetic energy can be shown to have the form

$$T = \frac{1}{2}\int_m \dot{\underline{R}}_d\cdot\dot{\underline{R}}_d\,dm = \frac{1}{2}\,m\,\dot{\underline{R}}_c\cdot\dot{\underline{R}}_c + \frac{1}{2}\,\underline{\omega}\cdot\underline{J}_d\cdot\underline{\omega}$$

$$+ \left(\underline{\omega}\times\int_m (\underline{r} + \underline{u}_c)\right)\cdot\dot{\underline{u}}_c'\,dm + \frac{1}{2}\int_m \dot{\underline{u}}_c'\cdot\dot{\underline{u}}_c'\,dm \qquad (1)$$

where $\underline{J}_d$ is the inertia dyadic of the deformed body about axes $\xi\eta\zeta$. The elements of the dyadic are

$$J_{\xi\xi}=\int_m[(y+v_c)^2+(z+w_c)^2]dm\ ,\ J_{\xi\eta}=J_{\eta\xi}=\int_m(x+u_c)(y+v_c)dm$$

$$J_{\eta\eta}=\int_m[(x+u_c)^2+(z+w_c)^2]dm\ ,\ J_{\xi\zeta}=J_{\zeta\xi}=\int_m(x+u_c)(z+w_c)dm \tag{2}$$

$$J_{\zeta\zeta}=\int_m[(x+u_c)^2+(y+v_c)^2]dm\ ,\ J_{\eta\zeta}=J_{\zeta\eta}=\int_m(y+v_c)(z+w_c)dm$$

The kinetic energy can be written conveniently in terms of matrix notation. If $\{\dot{R}_c\}$ is the column matrix corresponding to $\underset{\sim}{R}_c$, $\{\omega\}$ the column matrix corresponding to $\underset{\sim}{\omega}$, and $[J]$ the symmetric matrix, whose elements are the elements of the dyadic $\underset{\sim}{J}_d$, then Eq.(1) can be rewritten in the form

$$T=\frac{1}{2}m\{\dot{R}_c\}^T\{\dot{R}_c\}+\frac{1}{2}\{\omega\}^T[J]\{\omega\}+\{K\}^T\{\omega\}+\frac{1}{2}\int_m(\dot{u}_c^2+\dot{v}_c^2+\dot{w}_c^2)dm \tag{3}$$

where $\{K\}$ is the column matrix with the elements

$$\begin{aligned}K_\xi&=\int_m[(y+v_c)\dot{w}_c-(z+w_c)\dot{v}_c]\ dm\\K_\eta&=\int_m[(z+w_c)\dot{u}_c-(x+u_c)\dot{w}_c]\ dm\\K_\zeta&=\int_m[(x+u_c)\dot{v}_c-(y+v_c)\dot{u}_c]\ dm\end{aligned} \tag{4}$$

The angular velocity components $\omega_\xi,\omega_\eta,\omega_\zeta$ do not represent time rates of change of certain angles but nonintegrable combinations of time derivatives of angular displacements. They are sometimes referred to as time derivatives of quasi-coordinates. Denoting by θ_i and $\dot{\theta}_i (i=1,2,3)$ the true angular displacements and their time rates of change, the angular velocity vector can be written in the matrix form $\{\omega\} = [\theta]\{\dot{\theta}\}$, where $\{\dot{\theta}\}$ is the column matrix with elements $\dot{\theta}_i$ $(i=1,2,3)$ and $[\theta]$ is a 3x3 matrix, whose elements depend on the order of the three rotations θ_i used to produce the orientation of axes $\xi\eta\zeta$ relative to an inertial space. In view of this, the kinetic energy can be written in terms of true angular velocities as follows

$$T = \frac{1}{2}m\{\dot{R}_c\}^T\{\dot{R}_c\} + \frac{1}{2}\{\dot{\theta}\}^T[I]\{\dot{\theta}\} + \{L\}^T\{\dot{\theta}\} + \frac{1}{2}\int_m (\dot{u}_c^2+\dot{v}_c^2+\dot{w}_c^2)\,dm \qquad (5)$$

in which the notation

$$[I] = [\theta]^T[J][\theta] \quad , \quad \{L\} = [\theta]^T\{K\} \qquad (6)$$

has been adopted.

The potential energy arises primarily from two sources, namely gravity and body elasticity. The gravitational potential energy is assumed to be very small compared with the kinetic energy, or the elastic potential energy, and will be ignored. The elastic potential energy, denoted by V_{EL} and referred to at times as strain energy, depends on the nature of the elastic members and is in general a function of the partial derivatives of the elastic displacements u,v,w with respect to the spatial variables x,y,z. Since u_c,v_c,w_c differ from u,v,w by x_c,y_c,z_c, respectively, where the latter are independent of the spatial variables, V_{EL} can be regarded as depending on the partial derivatives of u_c,v_c,w_c with respect to x,y,z. We assume that V_{EL} is a function of spatial derivatives through second order but this assumption in no way affects the generality of the formulation. This particular functional dependence of V_{EL} should not be regarded as a restriction on the problem formulation, as the final formulation is expressed in a form which involves the partial derivatives only implicitly.

The system differential equations can be obtained by means of Hamilton's principle. To this end, a brief discussion of the generalized coordinates is in order. The motion of the mass center c is generally assumed not to be affected by the motion relative to c, so that it is possible to solve for the motion of c independently of the motion relative to c. As a result, the motion of c, referred to as orbital motion, can be regarded as known. We shall confine ourselves to the case in which the first term on the right side of Eq.(5) reduces to a known constant, so that the term can be ignored. This is clearly the case when the orbit is circular, or the motion of c is uniform or zero. It follows that the system generalized coordinates are the three rotations $\theta_i(t)$ and the three elastic displacements $u_c(x,y,z,t)$, $v_c(x,y,z,t)$, $w_c(x,y,z,t)$. The elastic displacements are defined only throughout the domain D_e, namely the subdomain of D corresponding to the elastic continuum, where D is a three-dimensional domain corresponding to the entire body. The domain D_e is bounded by the surface S.

For the holonomic system at hand, Hamilton's principle has the form

$$\delta \int_{t_1}^{t_2} L \, dt = 0 \tag{7}$$

where the motion must be such that the end conditions

$$\delta\theta_1 = \delta\theta_2 = \delta\theta_3 = \delta u_c = \delta v_c = \delta w_c = 0 \text{ at } t = t_1, t_2 \tag{8}$$

are satisfied. The integrand L in (7) is the Lagrangian which has the general functional form

$$L = T - V_{EL} = \int_D \hat{L}\left(\theta_i, \dot{\theta}_i, u_c, v_c, ---, \dot{w}_c, \frac{\partial u_c}{\partial x}, \frac{\partial u_c}{\partial y}, ---, \frac{\partial w_c}{\partial z}, \frac{\partial^2 u_c}{\partial x^2}, \frac{\partial^2 u_c}{\partial x \partial y}, ---, \frac{\partial^2 w_c}{\partial z^2}\right) dD \tag{9}$$

in which $\hat{L}$ is the Lagrangian density.

An application of Hamilton's principle leads to the system Lagrangian equations of motion. To this end, we consider Eq. (9) and write the variation of L as follows

$$\begin{aligned}\delta L = \int_D \Big[& \sum_{i=1}^{3} \left(\frac{\partial \hat{L}}{\partial \theta_i} \delta\theta_i + \frac{\partial \hat{L}}{\partial \dot{\theta}_i} \delta\dot{\theta}_i\right) + \frac{\partial \hat{L}}{\partial u_c} \delta u_c + \frac{\partial \hat{L}}{\partial v_c} \delta v_c + --- + \frac{\partial \hat{L}}{\partial \dot{w}_c} \delta \dot{w}_c \\ & + \frac{\partial \hat{L}}{\partial(\partial u_c/\partial x)} \delta\left(\frac{\partial u_c}{\partial x}\right) + \frac{\partial \hat{L}}{\partial(\partial u_c/\partial y)} \delta\left(\frac{\partial u_c}{\partial y}\right) + --- + \frac{\partial \hat{L}}{\partial(\partial w_c/\partial z)} \delta\left(\frac{\partial w_c}{\partial z}\right) \\ & + \frac{\partial \hat{L}}{\partial(\partial^2 u_c/\partial x^2)} \delta\left(\frac{\partial^2 u_c}{\partial x^2}\right) + \frac{\partial \hat{L}}{\partial(\partial^2 u_c/\partial x \partial y)} \delta\left(\frac{\partial^2 u_c}{\partial x \partial y}\right) + --- \\ & + \frac{\partial \hat{L}}{\partial(\partial^2 w_c/\partial z^2)} \delta\left(\frac{\partial^2 w_c}{\partial z^2}\right)\Big] \, dD\end{aligned} \tag{10}$$

Assuming that the functions u_c, v_c, w_c are well-behaved, we can interchange the variation and differentiation processes so that, after a series of integrations by parts with respect to the spatial variables, we arrive at

$$\int_D \Big[\frac{\partial\hat{L}}{\partial(\partial u_c/\partial x)}\,\delta\Big(\frac{\partial u_c}{\partial x}\Big)+\frac{\partial\hat{L}}{\partial(\partial u_c/\partial y)}\,\delta\Big(\frac{\partial u_c}{\partial y}\Big)+\cdots+\frac{\partial\hat{L}}{\partial(\partial w_c/\partial z)}\,\delta\Big(\frac{\partial w_c}{\partial z}\Big)$$

$$+\frac{\partial\hat{L}}{\partial(\partial^2 u_c/\partial x^2)}\,\delta\Big(\frac{\partial^2 u_c}{\partial x^2}\Big)+\frac{\partial\hat{L}}{\partial(\partial^2 u_c/\partial x\partial y)}\,\delta\Big(\frac{\partial^2 u_c}{\partial x\partial y}\Big)+\cdots$$

$$+\frac{\partial\hat{L}}{\partial(\partial^2 w_c/\partial z^2)}\,\delta\Big(\frac{\partial^2 w_c}{\partial z^2}\Big)\Big]dD=\int_{D_e}\underset{\sim}{\mathcal{L}}[u_c,v_c,w_c]\cdot\delta\underline{u}_c\,dD_e$$

$$+\underset{\sim}{B}_j[u_c,v_c,w_c]\cdot\underset{\sim}{B}_k[u_c,v_c,w_c]\Big|_S\quad,\ j=1,2;\ k=3,4 \tag{11}$$

where $\underset{\sim}{\mathcal{L}}(\mathcal{L}_{u_c},\mathcal{L}_{v_c},\mathcal{L}_{w_c})$ is a differential operator vector with components $\mathcal{L}_{u_c},\mathcal{L}_{v_c},\mathcal{L}_{w_c}$ defined over the domain D_e and $\underset{\sim}{B}_j(B_{ju_c},B_{jv_c},B_{jw_c})$, $\underset{\sim}{B}_k(B_{ku_c},B_{kv_c},B_{kw_c})$ are differential operator vectors defined at the surface S bounding the domain D_e, where the latter is recalled as being the domain within which the body possesses elasticity. We note, in passing, that in general if the components of $\underset{\sim}{\mathcal{L}}$ are of order 2p, where p is and integer, the ones of $\underset{\sim}{B}_j$ and $\underset{\sim}{B}_k$ are of order 2p-1 or less. Introducing Eqs. (10) and (11) into (7), integrating by parts with respect to time, and considering conditions (8), we obtain the ordinary differential equations for the attitude motion

$$\frac{\partial L}{\partial\theta_i}-\frac{d}{dt}\Big(\frac{\partial L}{\partial\dot{\theta}_i}\Big)=0\quad,\quad i=1,2,3 \tag{12}$$

and the partial differential equations for the elastic motion

$$\frac{\partial \hat{L}}{\partial u_c} - \frac{\partial}{\partial t}\left(\frac{\partial \hat{L}}{\partial \dot{u}_c}\right) + \mathcal{L}_{u_c}[u_c,v_c,w_c] = 0$$

$$\frac{\partial \hat{L}}{\partial v_c} - \frac{\partial}{\partial t}\left(\frac{\partial \hat{L}}{\partial \dot{v}_c}\right) + \mathcal{L}_{v_c}[u_c,v_c,w_c] = 0 \qquad (13)$$

$$\frac{\partial \hat{L}}{\partial w_c} - \frac{\partial}{\partial t}\left(\frac{\partial \hat{L}}{\partial \dot{w}_c}\right) + \mathcal{L}_{w_c}[u_c,v_c,w_c] = 0$$

where Eqs.(13) must be satisfied within the domain D_e. Moreover, the solutions of these equations must satisfy the boundary conditions

$$\underset{\sim}{B}_j[u_c,v_c,w_c] \cdot \underset{\sim}{B}_k[u_c,v_c,w_c] = 0 \text{ on } S \text{ , } j = 1,2;\ k = 3,4 \qquad (14)$$

We note that the motion of the system is described by a "hybrid" set of equations since Eqs.(12) are ordinary differential equations and Eqs.(13) are partial differential equations.

In any system in which elastic deformations take place there is certain damping present. We shall assume that the damping is internal and independent of the rotational motion of the body. We shall denote the components of the distributed damping forces by $\hat{Q}_{u_c}$, $\hat{Q}_{v_c}$, $\hat{Q}_{w_c}$ so that, whereas Eqs.(12) retain their form, Eqs.(13) become

$$\frac{\partial \hat{L}}{\partial u_c} - \frac{\partial}{\partial t}\left(\frac{\partial \hat{L}}{\partial \dot{u}_c}\right) + \mathcal{L}_{u_c}[u_c,v_c,w_c] + \hat{Q}_{u_c} = 0$$

$$\frac{\partial \hat{L}}{\partial v_c} - \frac{\partial}{\partial t}\left(\frac{\partial \hat{L}}{\partial \dot{v}_c}\right) + \mathcal{L}_{v_c}[u_c,v_c,w_c] + \hat{Q}_{v_c} = 0 \qquad (15)$$

$$\frac{\partial \hat{L}}{\partial w_c} - \frac{\partial}{\partial t}\left(\frac{\partial \hat{L}}{\partial \dot{w}_c}\right) + \mathcal{L}_{w_c}[u_c,v_c,w_c] + \hat{Q}_{w_c} = 0$$

The boundary conditions are not affected by damping so that they remain in the form (14).

Hamilton's Canonical Equations

We shall find it more convenient to work with a set of first-order Hamiltonian equations instead of the second-order Lagrangian equations. The order here relates to time and not spatial variables. To obtain the set of first-order differential equations, we consider the Hamiltonian defined by

$$H = \sum_{i=1}^{3} \frac{\partial L}{\partial \dot{\theta}_i} \dot{\theta}_i + \int_{D_e} \left(\frac{\partial \hat{L}}{\partial \dot{u}_c} \dot{u}_c + \frac{\partial \hat{L}}{\partial \dot{v}_c} \dot{v}_c + \frac{\partial \hat{L}}{\partial \dot{w}_c} \dot{w}_c \right) dD_e - L \tag{16}$$

and note that the Hamiltonian has a "hybrid" form, as it is both a function and a functional at the same time. Introducing the momenta

$$\begin{aligned} p_{\theta_i} &= \frac{\partial L}{\partial \dot{\theta}_i}, \quad i = 1,2,3 \\ \hat{p}_{u_c} &= \frac{\partial \hat{L}}{\partial \dot{u}_c}, \quad \hat{p}_{v_c} = \frac{\partial \hat{L}}{\partial \dot{v}_c}, \quad \hat{p}_{w_c} = \frac{\partial \hat{L}}{\partial \dot{w}_c} \end{aligned} \tag{17}$$

where the latter three are momentum densities, the Hamiltonian assumes the form

$$\begin{aligned} H &= \sum_{i=1}^{3} p_{\theta_i} \dot{\theta}_i + \int_{D_e} (\hat{p}_{u_c} \dot{u}_c + \hat{p}_{v_c} \dot{v}_c + \hat{p}_{w_c} \dot{w}_c) dD_e - L \\ &= \int_{D_e} \hat{H}\left(\theta_i, u_c, v_c, w_c, p_{\theta_i}, \hat{p}_{u_c}, \hat{p}_{v_c}, \hat{p}_{w_c}, \frac{\partial u_c}{\partial x}, \frac{\partial u_c}{\partial y}, ---, \frac{\partial w_c}{\partial z}, \right. \\ &\qquad \left. \frac{\partial^2 u_c}{\partial x^2}, \frac{\partial^2 u_c}{\partial x \partial y}, ---, \frac{\partial^2 w_c}{\partial z^2}\right) dD_e \end{aligned} \tag{18}$$

in which $\hat{H}$ is the Hamiltonian density. Considering both forms of H in (18), we can write the variation of the Hamiltonian as follows

$$\delta H = \sum_{i=1}^{3}(\delta p_{\theta_i}\dot{\theta}_i + p_{\theta_i}\delta\dot{\theta}_i) + \int_{D_e}(\delta\hat{p}_{u_c}\dot{u}_c + \hat{p}_{u_c}\delta\dot{u}_c + \cdots + \hat{p}_{w_c}\delta\dot{w}_c)dD_e$$

$$- \sum_{i=1}^{3}(\frac{\partial L}{\partial\theta_i}\delta\theta_i + \frac{\partial L}{\partial\dot{\theta}_i}\delta\dot{\theta}_i) - \int_{D_e}[\frac{\partial\hat{L}}{\partial u_c}\delta u_c + \frac{\partial\hat{L}}{\partial v_c}\delta v_c + \cdots + \frac{\partial\hat{L}}{\partial\dot{w}_c}\delta\dot{w}_c$$

$$+ \frac{\partial\hat{L}}{\partial(\partial u_c/\partial x)}\delta(\frac{\partial u_c}{\partial x}) + \frac{\partial\hat{L}}{\partial(\partial u_c/\partial y)}\delta(\frac{\partial u_c}{\partial y}) + \cdots + \frac{\partial\hat{L}}{\partial(\partial w_c/\partial z)}\delta(\frac{\partial w_c}{\partial z})$$

$$+ \frac{\partial\hat{L}}{\partial(\partial^2 u_c/\partial x^2)}\delta(\frac{\partial^2 u_c}{\partial x^2}) + \frac{\partial\hat{L}}{\partial(\partial^2 u_c/\partial x\partial y)}\delta(\frac{\partial^2 u_c}{\partial x\partial y}) + \cdots$$

$$+ \frac{\partial\hat{L}}{\partial(\partial^2 w_c/\partial z^2)}\delta(\frac{\partial^2 w_c}{\partial z^2})]dD_e = \sum_{i=1}^{3}(\frac{\partial H}{\partial\theta_i}\delta\theta_i + \frac{\partial H}{\partial p_{\theta_i}}\delta p_{\theta_i})$$

$$+ \int_{D_e}[\frac{\partial\hat{H}}{\partial u_c}\delta u_c + \frac{\partial\hat{H}}{\partial v_c}\delta v_c + \cdots + \frac{\partial\hat{H}}{\partial\hat{p}_{w_c}}\delta\hat{p}_{w_c} + \frac{\partial\hat{H}}{\partial(\partial u_c/\partial x)}\delta(\frac{\partial u_c}{\partial x})$$

$$+ \frac{\partial\hat{H}}{\partial(\partial u_c/\partial y)}\delta(\frac{\partial u_c}{\partial y}) + \cdots + \frac{\partial\hat{H}}{(\partial w_c/\partial z)}\delta(\frac{\partial w_c}{\partial z}) + \frac{\partial\hat{H}}{\partial(\partial^2 u_c/\partial x^2)}\delta(\frac{\partial^2 u_c}{\partial x^2}).$$

$$+ \frac{\partial\hat{H}}{\partial(\partial^2 u_c/\partial x\partial y)}\delta(\frac{\partial^2 u_c}{\partial x\partial y}) + \cdots + \frac{\partial\hat{H}}{\partial(\partial^2 w_c/\partial z^2)}\delta(\frac{\partial^2 w_c}{\partial z^2})]dD_e \quad (19)$$

Recalling definitions (17) and comparing coefficients of like variations in both forms of (19), we obtain the Hamiltonian equations

$$\dot{\theta}_i = \frac{\partial H}{\partial p_{\theta_i}}, \quad \dot{p}_{\theta_i} = -\frac{\partial H}{\partial\theta_i}, \quad i = 1,2,3 \quad (20a)$$

$$\dot{u}_c = \frac{\partial \hat{H}}{\partial \hat{p}_{u_c}} \quad , \quad \dot{v}_c = \frac{\partial \hat{H}}{\partial \hat{p}_{v_c}} \quad , \quad \dot{w}_c = \frac{\partial \hat{H}}{\partial \hat{p}_{w_c}}$$

$$\dot{\hat{p}}_{u_c} = - \frac{\partial \hat{H}}{\partial u_c} + \mathcal{L}_{u_c}[u_c, v_c, w_c] + \hat{Q}_{u_c}$$

$$\dot{\hat{p}}_{v_c} = - \frac{\partial \hat{H}}{\partial v_c} + \mathcal{L}_{v_c}[u_c, v_c, w_c] + \hat{Q}_{v_c} \tag{20b}$$

$$\dot{\hat{p}}_{w_c} = - \frac{\partial \hat{H}}{\partial w_c} + \mathcal{L}_{w_c}[u_c, v_c, w_c] + \hat{Q}_{w_c}$$

where Eqs.(20b) must be satisfied at every point of D_e. Note that to obtain the second half of Eqs.(20a) and (20b) use has been made of Lagrange's equations, Eqs.(12) and (15). Of course, the boundary conditions, Eqs.(14), remain the same.

When the kinetic energy is quadratic in the generalized velocities, the Hamiltonian reduces to the form

$$H = T + V_{EL} \tag{21}$$

which is recognized as the system total energy.

Stability of Hybrid Dynamical Systems

The motion of an n-degree-of-freedom dynamical system can be described by 2n first-order differential equations of motion, namely, Hamilton's equations. The state of the system of any time t is given by the 2n canonical variables $q_k(t)$, $p_k(t)$ $(k = 1,2,---, n)$, where q_k are generalized coordinates and p_k generalized conjugate momenta. For a given set of initial conditions, the state of the system can be represented by a vector

$\mathbf{x}(t)$ in a 2n-dimensional vector space, known as the phase space. The Liapunov definition of stability places certain restrictions on the norm $\|\mathbf{x}(t)\|$. In particular, the trivial solution is stable if for any arbitrary $\varepsilon > 0$ and time t_0 there is a number $\delta(\varepsilon, t_0) > 0$ such that if the inequality $\|\mathbf{x}_0\| < \delta$ is satisfied at t_0, then the inequality $\|\mathbf{x}(t)\| < \varepsilon$ is satisfied for all $t \geq t_0$. From the preceding discussion it is clear that the stability definition for a discrete system places restrictions on the generalized coordinates and momenta q_k, p_k, or alternatively on the generalized coordinates and velocities q_k, $\dot{q}_k (k = 1,2,\text{---}, n)$.

In the case of distributed systems the generalized coordinates depend not only on time but also on spatial coordinates. The displacement vector at any point P with spatial coordinates x,y,z, so that $P = P(x,y,z)$, and at any time t is given by $\mathbf{u} = \mathbf{u}(P,t)$, where $\mathbf{u}$ is a vector with components $u(P,t)$, $v(P,t)$, and $w(P,t)$ along x,y, and z, respectively. We shall be concerned in this paper exclusively with cases in which a small initial state ensures a small initial potential energy.

Next let us consider a hybrid system with the state vector given by $\mathbf{v} = \mathbf{v}_d(t) + \mathbf{v}_c(P,t)$, where $\mathbf{v}_d(t)$ and $\mathbf{v}_c(P,t)$ represent discrete and continuous variables, respectively. The system is described by the set of differential equations

$$\dot{\mathbf{v}} = \mathbf{V}(\mathbf{v}, \partial\mathbf{v}_c/\partial x, \partial\mathbf{v}_c/\partial y, \text{---}, \partial^{2p}\mathbf{v}_c/\partial z^{2p}) \qquad (22)$$

where $\underset{\sim}{V}$ is a vector function depending on the state vector and spatial derivatives of the state vector through order 2p, in which p is an integer. The continuous variables must also satisfy appropriate boundary conditions. The state vector can be imagined geometrically as representing an element in a space S which can be regarded as the cartesian product of a finite dimensional vector space and a function space, the first corresponding to $\underset{\sim}{v}_d$ and the second associated with $\underset{\sim}{v}_c$. The motion of the system can be interpreted as a continuous mapping of the space S onto itself, which implies that if the state of the system at a given time is known, then the state is known at any subsequent time. A solution of system (22) constant in time, namely, a set of constants satisfying

$$\underset{\sim}{V}(\underset{\sim}{v}, \partial \underset{\sim}{v}_c/\partial x \; , \; \partial \underset{\sim}{v}_c/\partial y, \; --- \; , \; \partial^{2p} \underset{\sim}{v}_c/\partial z^{2p}) = \underline{0} \tag{23}$$

is known as a singular point or equilibrium point. We shall be interested in the stability of motion in the neighborhood of equilibrium points. Assuming, without loss of generality, that the origin of S is an equilibrium point, we shall concern ourselves with the stability of the trivial solution, known also as the null solution.

Stability is now defined in a manner analogous to the Liapunov definitions of stability for discrete systems. To this end, we first introduce the norm $\|\underset{\sim}{v}(t)\| = \|\underset{\sim}{v}_d(t)\| + \int_D \|\underset{\sim}{v}_c(P,t)\| \, dD(P)$, where D is the domain over which continuous variables are defined, and denote by $\|\underset{\sim}{v}_0\|$ the value of the norm at $t = t_0$. Then the trivial solution is defined as stable if for any arbitrary positive quantity ε and time t_0 there exists a positive number $\delta(\varepsilon, t_0)$ such that the satisfaction of the inequality $\|\underset{\sim}{v}_0\| < \delta$ implies the satisfaction of the inequality $\|\underset{\sim}{v}(t)\| < \varepsilon$ for all $t \geq t_0$. If, in addition, $\lim_{t \to \infty} \|\underset{\sim}{v}(t)\| = 0$, then the trivial solution is asymptotically stable. It is stressed again that we are concerned exclusively with the cases in which a small initial state $\|\underset{\sim}{v}_0\|$ implies also small spatial derivatives, at least through order p. The trivial solution is unstable if it is not stable.

To test the stability of system (22) in the neighborhood of the trivial solution, we define a scalar functional $U = U(\underset{\sim}{v}, \partial\underset{\sim}{v}_c/\partial x, \partial\underset{\sim}{v}_c/\partial y, ---, \partial^p\underset{\sim}{v}_c/\partial z^p)$ such that $U(\underset{\sim}{0}, \underset{\sim}{0}, ---, \underset{\sim}{0}) = 0$. Actually U is both a function and a functional simultaneously but we shall call it a functional. We note that U depends on spatial derivatives through order p, as opposed to $\underset{\sim}{V}$ which depends on derivatives through order 2p. Moreover, the total time derivative of U along a trajectory of the system is defined by $\dot{U} = dU/dt = \underset{\sim}{\nabla}U_d \cdot \dot{\underset{\sim}{v}}_d + \int_D \underset{\sim}{\nabla}\hat{U}_c \cdot \dot{\underset{\sim}{v}}_c dD = \underset{\sim}{\nabla}U_d \cdot \underset{\sim}{V}_d + \int_D \underset{\sim}{\nabla}\hat{U}_c \cdot \underset{\sim}{V}_c dD$, where the subscripts d and c designate quantities pertaining to discrete and continuous variables, respectively.

At this point we consider the following theorems:

Theorem 1 - If there exists for system (22) a positive (negative) definite functional U whose total time derivative $\dot{U}$ is negative (positive) semidefinite along every trajectory of (22), then the trivial solution is stable.

Theorem 2 - If the conditions of Theorem 1 are satisfied and if in addition the set of points at which $\dot{U}$ is zero contains no nontrivial positive half-trajectory, then the trivial solution is asymptotically stable.

Theorem 3 - If there exists for system (22) a functional U whose total time derivative $\dot{U}$ is positive (negative)definite along every trajectory of (22) and the function itself can assume positive (negative) values in the neighborhood of the origin, then the trivial solution is unstable.

Theorem 4 - Suppose that a functional U such as in Theorem 3 exists but for which $\dot{U}$ is only positive (negative) semidefinite and, in addition, the set of points at which $\dot{U}$ is zero contains no nontrivial positive half-trajectory. Suppose that in every neighborhood of the origin there is a point $\underset{\sim}{v}_0$ such that for arbitrary $t_0 > 0$ we have $U\big|_{\underset{\sim}{v}=\underset{\sim}{v}_0} > 0 (< 0)$. Then the trivial solution is unstable and the trajectories $\underset{\sim}{v}(\underset{\sim}{v}_0, t_0, t)$ for which $U\big|_{\underset{\sim}{v}=\underset{\sim}{v}_0} > 0 (< 0)$ must leave the open domain $\|\underset{\sim}{v}\| < \varepsilon$ as the time t increases.

A testing functional U satisfying any of the preceding theorems is referred to as a Liapunov functional. Theorems 1 and 3 are associated with the name of Liapunov, whereas Theorems 2 and 4 with that of Krasovskii. A discussion of these theorems for discrete systems can be found in Ref. 1 (Sec. 6.7).

A functional is defined as positive (negative) definite if it is never negative (positive) and it is zero only if $\underset{\sim}{v}$ is identically zero. Continuous variables must be zero over the entire domain D. A functional is said to be positive (negative) semidefinite if it is never negative (positive) but can be zero at points other than the origin.

Since the scalar functional U depends on spatial derivatives of $\underset{\sim}{v}$, it may be difficult at times to determine its sign properties. In such cases it may be possible to define another scalar functional $W(\underset{\sim}{v})$, depending on the state vector $\underset{\sim}{v}$ alone, and such that $U \geq W$. Then we can state the following:

Stability Theorem - Suppose that for system (22) there exists a scalar functional U such that $\dot{U}$ is negative semidefinite along every trajectory of (22) and, in addition, the set of points at which $\dot{U}$ is zero contains no nontrivial positive half-trajectory. Then, if a positive definite functional W can be found such that $U \geq W$, the trivial solution $\underset{\sim}{v} = \underset{\sim}{0}$ is asymptotically stable.

The above Stability Theorem has significant implications as far as the stability analysis of hybrid dynamical systems of the type considered here is concerned.

The Hamiltonian as a Liapunov Functional

We shall show next that under certain circumstances the Hamiltonian can be used as a Liapunov functional. Taking the total time derivative of H from the first form of Eq.(18) and using Eqs.(12)and (15), as well as boundary conditions (14) and definitions (17), we obtain

$$\dot{H} = \int_{D_e} (\hat{Q}_{u_c} \dot{u}_c + \hat{Q}_{v_c} \dot{v}_c + \hat{Q}_{w_c} \dot{w}_c) dD_e \tag{24}$$

Next we assume that the damping forces are such that $\dot{H}$ is negative semidefinite.

$$\dot{H} \leq 0 \tag{25}$$

Moreover, due to coupling, the forces $\hat{Q}_{u_c}$, $\hat{Q}_{v_c}$, $\hat{Q}_{w_c}$ are never identically zero at every point of the phase space but they reduce to zero at an equilibrium point. Hence, if the Hamiltonian H is positive definite at an equilibrium point, then by Theorem 2, H can be regarded as a Liapunov functional and the equilibrium point under consideration as asymptotically stable. On the other hand, if H is not positive definite and there are points for which it is negative, then by Theorem 4 the equilibrium point is unstable.

In view of the preceding discussion, we shall consider the Hamiltonian as a Liapunov functional. As indicated by Eq.(23), the equilibrium positions are thos rendering the right sides of Eqs.(20) equal to zero. Hence, the equilibrium positions are the solutions of the equations

$$\frac{\partial H}{\partial p_{\theta_i}} = 0 \quad , \quad -\frac{\partial H}{\partial \theta_i} = 0 \quad , \quad i = 1,2,3 \tag{26a}$$

$$\frac{\partial \hat{H}}{\partial \hat{p}_{u_c}} = \frac{\partial \hat{H}}{\partial \hat{p}_{v_c}} = \frac{\partial \hat{H}}{\partial \hat{p}_{w_c}} = 0$$

$$-\frac{\partial \hat{H}}{\partial u_c} + \mathcal{L}_{u_c}[u_c, v_c, w_c] = 0$$

$$-\frac{\partial \hat{H}}{\partial v_c} + \mathcal{L}_{v_c}[u_c, v_c, w_c] = 0 \tag{26b}$$

$$-\frac{\partial \hat{H}}{\partial w_c} + \mathcal{L}_{w_c}[u_c, v_c, w_c] = 0$$

where Eqs.(26b) must be satisfied at every point of D_e.

From Eq.(21) we see that for a conservative system the Hamiltonian can be expressed as the sum of the kinetic and potential energies, where the kinetic energy is given by Eq.(3). The elastic potential energy depends upon the type of system considered, but is in general a function of the elastic displacements u,v,w, and spatial derivatives of these displacements. If we assume that the elastic displacements are independent of one another, V_{EL} can be shown to reduce to

$$V_{EL} = \frac{1}{2} \int_{D_e} (u \mathcal{L}_u[u] + v \mathcal{L}_v[v] + w \mathcal{L}_w[w]) dD_e \tag{27}$$

where, assuming that the differential operators $\mathcal{L}_u$, $\mathcal{L}_v$, and $\mathcal{L}_w$ are of order four, the elastic displacements u,v, and w are subject to the boundary conditions

$$\begin{aligned} &B_{ju}[u] = 0 \quad \text{or} \quad B_{ku}[u] = 0 \\ &B_{jv}[v] = 0 \quad \text{or} \quad B_{kv}[v] = 0 \quad \text{on } S,\ j = 1,2;\ k = 3,4 \\ &B_{jw}[w] = 0 \quad \text{or} \quad B_{kw}[w] = 0 \end{aligned} \tag{28}$$

Under these conditions, the eigenvalue problem corresponding to the elastic motion separates into three individual eigenvalue problems defined by the differential equations

$$\mathcal{L}_u[u] = \Lambda_u^2 M_u[u] \ , \mathcal{L}_v[v] = \Lambda_v^2 M_v[v] \ , \mathcal{L}_w[w] = \Lambda_w^2 M_w[w] \tag{29}$$

which must be satisfied over the domain D_e and by the boundary conditions (28), respectively.

At this point let us define the Rayleigh quotient associated with u as follows

$$R_u(u) = \frac{\int_{D_e} u\mathcal{L}_u[u]\, dD_e}{\int_{D_e} u\, M_u[u]\, dD_e} \tag{30}$$

For positive definite operators $\mathcal{L}_u$ and M_u the quotient $R_u(u)$ is always positive. Moreover, denoting by Λ_{u1}^2 the lowest eigenvalue associated with the vibration u, it can be shown that (see Ref. 17, Sec. 5-14)

$$R_u(u) \geq \Lambda_{u1}^2 \tag{31}$$

Analogous statements can be made in regard to the displacements v and w. It follows from (30) and (31), together with similar expressions for v and w, that

$$\begin{aligned} V_{EL} &= \frac{1}{2} \int_{D_e} (u\mathcal{L}_u[u] + v\mathcal{L}_v[v] + w\mathcal{L}_w[w])\, dD_e \\ &\geq \frac{1}{2} \int_{D_e} \rho(\Lambda_{u1}^2 u^2 + \Lambda_{v1}^2 v^2 + \Lambda_{w1}^2 w^2)\, dD_e \end{aligned} \tag{32}$$

where the operators M_u, M_v, and M_w in this case turn out to be merely the mass density ρ. If in addition the displacements u_c, v_c, and w_c (which differ from u, v, and w by rigid-body translations x_c, y_c, and z_c) are independent, it is not difficult to show that

$$V_{EL} \geq \frac{1}{2} \int_{D_e} \rho(\Lambda_{u1}^2 u_c^2 + \Lambda_{v1}^2 v_c^2 + \Lambda_{w1}^2 w_c^2)\, dD_e \tag{33}$$

However, if u_c, v_c, and w_c are coupled through the center of mass motion Eq.(33) does not hold in general. We shall be concerned with cases where Eq.(32) is valid but not Eq.(33).

Let us define a functional κ as follows

$$\kappa = T + \frac{1}{2} \int_{D_e} \rho (\Lambda_{u1}^2 u^2 + \Lambda_{v1}^2 v^2 + \Lambda_{w1}^2 w^2) dD_e \tag{34}$$

It follows from Eqs.(21) and (32) that

$$H \geq \kappa \tag{35}$$

Hence, from our Stability Theorem the equilibrium solution is asymptotically stable if κ is positive definite.

Torque-Free Systems

When there are no motion integrals, the state at time t of the hybrid system considered is given by an element in a space S which can be regarded as the cartesian product of the finite dimensional vector space defined by θ_i, p_{θ_i} $(i=1,2,3)$ and the function space defined by u_c, v_c, w_c, $\hat{p}_{u_c}$, $\hat{p}_{v_c}$, $\hat{p}_{w_c}$. The space S is simply the phase space. Alternatively, the space can be regarded as the cartesian product of the vector space defined by θ_i, $\dot{\theta}_i$ $(i=1,2,3)$ and the function space defined by u_c, v_c, w_c, $\dot{u}_c$, $\dot{v}_c$, $\dot{w}_c$. The motion of the system can be interpreted as a continuous mapping of the space S onto itself. This implies that if the state of the system at a given time is known, then the state is known for any subsequent time.

Under certain circumstances the system possesses motion integrals. For example, such integrals occur when the system is free of external torques, in which case the motion integrals are simply momentum integrals. These integrals can be regarded as constraint equations relating the system velocities. Constraints may be interpreted as restricting the motion to a subspace of a correspondingly smaller dimension.

Let us assume that the system considered is free of external forces, so that the three torque components about the mass center c are zero. It follows that the angular momentum vector about c is conserved

$$\underset{\sim}{L}_c = \int_m (\underset{\sim}{r}+\underset{\sim}{u}_c)\times[\dot{\underset{\sim}{u}}'_c + \underset{\sim}{\omega}\times(\underset{\sim}{r}+\underset{\sim}{u}_c)]\ dm = \underset{\sim}{\beta} = \text{const} \tag{36}$$

in which $\underset{\sim}{\beta}$ denotes the constant angular momentum vector. In matrix notation, Eq.(36) assumes the form

$$[J]\{\omega\} + \{K\} = \{\beta\} \tag{37}$$

where [J] is the inertia matrix of the deformed body, namely, the matrix representation of the inertia dyadic whose elements are given by Eqs.(2), and {K} is the column matrix of the angular momentum components due to the elastic motion; the elements of {K} are given by Eqs.(4). Clearly, $\{\beta\}$ is the matrix representation of the vector $\underset{\sim}{\beta}$.

Equation (37) can be used to eliminate the angular velocities $\dot{\theta}_i\,(i=1,2,3)$ from the kinetic energy. Indeed, premultiplying Eq.(37) by $[J]^{-1}$ and rearranging, we obtain

$$\{\omega\} = [J]^{-1}\{\beta - K\} \tag{38}$$

Introducing Eq.(38) into Eq.(3), and ignoring the term due to the orbital motion, we can write the kinetic energy in the form

$$T = T_2 + T_0 \tag{39}$$

in which

$$T_2 = \frac{1}{2} \int_m \{\dot{u}_c'\}^T \{\dot{u}_c'\} \, dm - \frac{1}{2} \{K\}^T [J]^{-1} \{K\} \tag{40}$$

is a quadratic expression in the elastic velocities $\dot{u}_c$, $\dot{v}_c$, $\dot{w}_c$, and

$$T_0 = \frac{1}{2} \{\beta\}^T [J]^{-1} \{\beta\} \tag{41}$$

is an expression in the angular coordinates and elastic displacements alone, hence it contains no velocities. It turns out that not all three angular coordinates are present in T_0 but only two of them. To show this, we denote by β_0 the magnitude of the initial angular momentum vector, assume for convenience that the direction of the angular momentum vector coincides initially with the inertial axis Z, and express the angular momentum matrix $\{\beta\}$ in the form $\beta_0\{\ell\}$, where $\{\ell\}$ is the column matrix of the direction cosines $\ell_{\xi Z}$, $\ell_{\eta Z}$, $\ell_{\zeta Z}$ between Z and axes ξ, η, ζ, respectively. These direction cosines can be expressed in terms of only two angular coordinates.

Inserting Eq.(39), in conjunction with expressions (40) and (41), into Eq.(34), we conclude that the functional κ can be written in the form

$$\kappa = \kappa_1 + \kappa_2 \tag{42}$$

in which $\kappa_1 = T_2$ and

$$\begin{aligned} \kappa_2 &= T_0 + \frac{1}{2} \int_{D_e} \rho (\Lambda_{u1}^2 u^2 + \Lambda_{v1}^2 v^2 + \Lambda_{w1}^2 w^2) \, dD_e \\ &= \frac{1}{2} \beta_0^2 \{\ell\}^T [J]^{-1} \{\ell\} + \frac{1}{2} \int_{D_e} \rho \{u\}^T \lceil \Lambda_1^2 \rfloor \{u\} \, dD_e \end{aligned} \tag{43}$$

where $\{u\}$ is the column matrix of the elastic displacements u, v, w and $\lceil\Lambda_{I}^{2}\rfloor$ is a diagonal matrix of the lowest eigenvalues associated with these displacements. The functional κ_2 can be regarded as a modified dynamic potential. By virtue of inequality (32), we conclude that κ_2 is in general smaller than (or equal to) the ordinary dynamic potential $T_0 + V_{EL}$.

Since κ can be written as the sum of κ_1 and κ_2, where κ_1 is a quadratic functional in the generalized velocities, and κ_2 depends only on the generalized coordinates, κ is positive definite if and only if κ_1 and κ_2 are both positive definite. By definition the quadratic part of the kinetic energy, T_2, is positive definite, so that we conclude that if κ_2 is positive definite κ is positive definite.

To obtain the testing functional κ_2, we recall that the elements of the inertia matrix [J] of the deformed body are given by Eqs.(2). It is not difficult to show that the matrix [J] can be written as the sum of two matrices $[J]_0$ and $[J]_1$ where $[J]_0$ denotes the inertia matrix about axis x,y,z of the body in undeformed state which really represents the matrix of principal moments of inertia for the undeformed body. Matrix $[J]_1$ represents the change in the inertia matrix due to the elastic displacements about axes ξ,η,ζ as well as the change in the inertia matrix of the undeformed body due to the translations x_c, y_c, z_c, of the origin. Since the elastic displacements u_c, v_c, w_c, as well as the coordinates x_c, y_c, z_c, of the center of mass are assumed small, the matrix $[J]_1$ is small compared to $[J]_0$. Hence, writing the matrix $[J]$ as

$$[J] = [J]_0 + [J]_1 \tag{44}$$

because $[J]_1$ is small compared to $[J]_0$, it is not difficult to show that

$$[K]=[J]^{-1}=[J]_0^{-1}-[J]_0^{-1}[J]_1[J]_0^{-1}+[J]_0^{-1}[J]_1[J]_0^{-1}[J]_1[J]_0^{-1} \tag{45}$$

where $[K]$ denotes the inverse of $[J]$. We may therefore express our testing function in the form

$$\kappa_2 = \frac{1}{2}\,\beta_0^2\{\ell\}^T[K]\{\ell\} + \frac{1}{2}\int_{D_e}\{u\}^T\lceil\Lambda_1^2\rfloor\{u\}dD_e \tag{46}$$

where $[K]$ is given by Eq.(45).

The problem of investigating stability reduces to that of testing expression (46) for sign definiteness. To this end, we expand κ_2 in the neighborhood of an equilibrium point E and ignore terms of order greater than two. This process leaves us with a quadratic expression, denoted by $\kappa_2|_E$, in the generalized coordinates. However, the generalized coordinates representing the elastic displacements appear in integrals defined over the elastic domain, which precludes its testing for sign definiteness by standard means. This problem can be circumvented through the use of modal analysis in conjunction with series truncation. To this end, we must solve the eigenvalue problems associated with the elastic displacements u, v, w, and represent these displacements by finite series of corresponding eigenfunctions multiplying associated generalized coordinates, where the first depend on spatial coordinates alone and the latter on time alone. Now we are in the position to perform integrations with respect to the spatial variables and write $\kappa_2|_E$ as a quadratic form in the newly defined generalized coordinates. We can define the Hessian matrix $[\mathcal{H}]_E$ corresponding to this quadratic expression, and it should be noted that the order of the Hessian matrix depends on the number of eigenfunctions used in the series representing the elastic displacements. The sign definiteness of $[\mathcal{H}]_E$ may be ascertained by means of Sylvester's criterion (see Ref. 1, Sec. 6.7). An alternative approach to testing the sign definiteness of $\kappa_2|_E$ involves defining new coordinates representing certain integrals appearing in $\kappa_2|_E$ and using Schwarz's inequality for for functions to discretize $\kappa_2|_E$. In general this procedure involves considerably less effort than using modal analysis and yields sharper stability criteria.

The Stability of High-Spin Motion of a Satellite with Flexible Appendages.

The general theory developed in the preceding sections will now be used to investigate the stability of a satellite simulated by a main rigid body and six flexible thin rods, as shown in Figure 2a. In the undeformed state the body possesses principal moments of inertia A,B,C about axes x,y,z, respectively, and the rods are aligned with these axes. The body is initially spinning undeformed about axis z with angular velocity

Ω_s. The domain of the elastic continuum D_e consist of three subdomains D_x, D_y, D_z, bounded by S_x, S_y, S_z, where

$$D_x : -(h_x+\ell_x) < x < -h_x,\ h_x < x < (h_x+\ell_x),\ S_x = \pm h_x,\ \pm (h_x+\ell_x)$$

$$D_y : -(h_y+\ell_y) < y < -h_y,\ h_y < y < (h_y+\ell_y),\ S_y = \pm h_y,\ \pm (h_y+\ell_y)$$

$$D_z : -(h_z+\ell_z) < z < -h_z,\ h_z < z < (h_z+\ell_z),\ S_z = \pm h_z,\ \pm (h_z+\ell_z)$$

Hence $\underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$ over $D-D_e$, $\underset{\sim}{r} = x\underset{\sim}{i}$ over D_x, $\underset{\sim}{r} = y\underset{\sim}{j}$ over D_y, and $\underset{\sim}{r} = z\underset{\sim}{k}$ over D_z. Assuming only flexural transverse vibrations, it follows that

$$\underset{\sim}{u} = \underset{\sim}{u}_x = v_x\underset{\sim}{j} + w_x\underset{\sim}{k},\ \underset{\sim}{u}_c = \underset{\sim}{u}_{cx} = v_{cx}\underset{\sim}{j} + w_{cx}\underset{\sim}{k},\ \underset{\sim}{r}_c = y_c\underset{\sim}{j} + z_c\underset{\sim}{k} \text{ over } D_x$$

$$\underset{\sim}{u} = \underset{\sim}{u}_y = u_y\underset{\sim}{i} + w_y\underset{\sim}{k},\ \underset{\sim}{u}_c = \underset{\sim}{u}_{cy} = u_{cy}\underset{\sim}{i} + w_{cy}\underset{\sim}{k},\ \underset{\sim}{r}_c = x_c\underset{\sim}{i} + z_c\underset{\sim}{k} \text{ over } D_y$$

$$\underset{\sim}{u} = \underset{\sim}{u}_z = u_z\underset{\sim}{i} + v_z\underset{\sim}{j},\ \underset{\sim}{u}_c = \underset{\sim}{u}_{cz} = u_{cz}\underset{\sim}{i} + v_{cz}\underset{\sim}{j},\ \underset{\sim}{r}_c = x_c\underset{\sim}{i} + y_c\underset{\sim}{j} \text{ over } D_z$$

From Eqs.(2) we conclude that the moments and products of inertia of the deformed body have the values

$$J_{\xi\xi} = A + \int_{D_x} \rho_x (v_{cx}^2+w_{cx}^2)\,dx + \int_{D_y} \rho_y w_{cy}^2\,dy + \int_{D_z} \rho_z v_{cz}^2\,dz + m(y_c^2+z_c^2)$$

$$J_{\eta\eta} = B + \int_{D_x} \rho_x w_{cx}^2\,dx + \int_{D_y} \rho_y (u_{cy}^2+w_{cy}^2)\,dy + \int_{D_z} \rho_z u_{cz}^2\,dz + m(x_c^2+z_c^2)$$

$$J_{\zeta\zeta} = C + \int_{D_x} \rho_x v_{cx}^2\,dx + \int_{D_y} \rho_y u_{cy}^2\,dy + \int_{D_z} \rho_z (u_{cz}^2+v_{cz}^2)\,dz + m(x_c^2+y_c^2) \tag{47}$$

$$J_{\xi\eta} = J_{\eta\xi} = \int_{D_x} \rho_x x v_{cx}\,dx + \int_{D_y} \rho_y y u_{cy}\,dy + \int_{D_z} \rho_z u_{cz} v_{cz}\,dz + m x_c y_c$$

$$J_{\xi\zeta}=J_{\zeta\xi}=\int_{D_x}\rho_x x w_{cx}dx+\int_{D_y}\rho_y u_{cy}w_{cy}dy+\int_{D_z}\rho_z z u_{cz}dz + m x_c z_c$$

$$J_{\eta\zeta}=J_{\zeta\eta}=\int_{D_x}\rho_x v_{cx}w_{cx}dx+\int_{D_y}\rho_y w_{cy}dy+\int_{D_z}\rho_z z v_{cz}dz + m y_c z_c$$

where ρ_x, ρ_y, ρ_z represent mass per unit length associated with the respective rods. We shall assume that the mass of the rods is symmetrically distributed, such that $\rho_x(-x)=\rho_x(x)$, $\rho_y(-y) = \rho_y(y)$, and $\rho_z(-z) = \rho_z(z)$. Examining the elements of [J], we conclude that

$$[J]_0 = \begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{bmatrix} \tag{48}$$

$$[J]_1 = \begin{bmatrix} J_{\xi\xi}-A & -J_{\xi\eta} & -J_{\xi\zeta} \\ -J_{\eta\xi} & J_{\eta\eta}-B & -J_{\eta\zeta} \\ -J_{\zeta\xi} & -J_{\zeta\eta} & J_{\zeta\zeta}-C \end{bmatrix}$$

We shall be interested in investigating the stability of the high-spin motion in which the undeformed satellite rotates with constant angular velocity Ω_s about axis z. Hence, we consider the stability in the neighborhood of the equilibrium point

$$\theta_1 = \theta_2 = u_y = u_z = v_x = v_z = w_x = w_y = 0 \tag{49}$$

which, in turn, implies that

$$u_{cy} = u_{cz} = v_{cx} = v_{cz} = w_{cx} = 0 \tag{50}$$

Since in the equilibrium configuration the body spins about axis z with angular velocity Ω_s, where z coincides with the inertial axis Z, it follows that $\beta_0 = C\Omega_s$. Moreover, from Fig. 2b we conclude that the direction cosines have the values $\ell_{\xi Z} = -\cos\theta_1 \sin\theta_2$, $\ell_{\eta Z} = \sin\theta_1$, and $\ell_{\zeta Z} = \cos\theta_1 \cos\theta_2$. Introducing all these values into the first term of Eq.(43), considering Eq. (45), and ignoring terms in $\theta_1, \theta_2, u_{cy}, u_{cz}, v_{cx}, v_{cz}, w_{cx}, w_{cy}, x_c, y_c$ and z_c of order larger than two (as well as constant terms), we can write

$$\beta_0^2\{\ell\}^T[K]\{\ell\}\Big|_E = \Omega_s^2[\frac{C}{B}(C-B)\theta_1^2 + \frac{C}{A}(C-A)\theta_2^2$$

$$- 2\frac{C}{A}\theta_2(\int_{D_x}\rho_x x w_{cx} dx + \int_{D_z}\rho_z z u_{cz} dz) + 2\frac{C}{B}\theta_1(\int_{D_y}\rho_y y w_{cy} dy$$

$$+ \int_{D_z}\rho_z z v_{cz} dz) - \int_{D_x}\rho_x v_{cx}^2\, dx - \int_{D_y}\rho_y u_{cy}^2\, dy$$

$$- \int_{D_z}\rho_z(u_{cz}^2 + v_{cz}^2)dz - m(x_c^2 + y_c^2) + \frac{1}{A}(\int_{D_x}\rho_x x w_{cx} dx$$

$$+ \int_{D_z}\rho_z z u_{cz} dz)^2 + \frac{1}{B}(\int_{D_y}\rho_y y w_{cy} dy + \int_{D_z}\rho_z z v_{cz} dz)^2] \tag{51}$$

Recalling our testing functional κ_2, as given in Eq.(43), we note that the second term, due to the elastic potential, involves the actual elastic displacements u_y, u_z, --- , w_y, whereas Eq.(51), representing the first term of κ_2, involves the displacements u_{cy}, u_{cz}, --- , w_{cy}, as well as the center of mass coordinates, x_c and y_c. For consistency, we will replace in Eq.(51) the displacements u_{cy}, u_{cz}, --- , w_{cy} by $u_y - x_c$, $u_z - x_c$, --- , $w_y - z_c$, respectively. To this end, we note that the quantities ρ_x, ρ_y and ρ_z are even functions of the spatial coordinates. Considering the definitions of the domains D_x, D_y and D_z, and recalling that x_c, y_c, z_c do not depend on spatial coordinates, it is not difficult to show that

$$\begin{aligned} \int_{D_z} \rho_z z\, u_{cz}\, dz &= \int_{D_z} \rho_z z\, u_z\, dz \\ \int_{D_z} \rho_z z\, v_{cz}\, dz &= \int_{D_z} \rho_z z\, v_z\, dz \\ \int_{D_x} \rho_x x\, w_{cx}\, dx &= \int_{D_x} \rho_x x\, w_x\, dx \\ \int_{D_y} \rho_y y\, w_{cy}\, dy &= \int_{D_y} \rho_y y\, w_y\, dy \end{aligned} \tag{52}$$

But the definitions of x_c and y_c are

$$\begin{aligned} x_c &= \frac{1}{m} \left(\int_{D_y} \rho_y u_y\, dy + \int_{D_z} \rho_z u_z\, dz \right) \\ y_c &= \frac{1}{m} \left(\int_{D_x} \rho_x v_x\, dx + \int_{D_z} \rho_z v_z\, dz \right) \end{aligned} \tag{53}$$

so that

$$\begin{aligned} \int_{D_y} \rho_y u_{cy}^2 dy + \int_{D_z} \rho_z u_{cz}^2 dz &= \int_{D_y} \rho_y u_y^2 dy + \int_{D_z} \rho_z u_z^2 dz - (2m - m_y - m_z) x_c^2 \\ \int_{D_x} \rho_x v_{cx}^2 dx + \int_{D_z} \rho_z v_{cz}^2 dz &= \int_{D_x} \rho_x v_x^2 dx + \int_{D_z} \rho_z v_z^2 dz - (2m - m_x - m_z) y_c^2 \end{aligned} \tag{54}$$

where $m_x = 2\int_{h_x}^{h_x+\ell_x} \rho_x\, dx$, $m_y = 2\int_{h_y}^{h_y+\ell_y} \rho_y\, dy$, $m_z = 2\int_{h_z}^{h_z+\ell_z} \rho_z\, dz$. Inserting Eqs.(52) and (54) into Eq.(51), we obtain

$$\beta_0^2\{\ell\}^T[K]\{\ell\}\Big|_E = \Omega_s^2\Big[\frac{C}{B}(C-B)\theta_1^2 + \frac{C}{A}(C-A)\theta_2^2 - 2\,\frac{C}{A}\,\theta_2\Big(\int_{D_x}\rho_x x w_x\, dx$$
$$+ \int_{D_z}\rho_z z u_z dz\Big) + 2\,\frac{C}{B}\,\theta_1\Big(\int_{D_y}\rho_y y w_y d_y + \int_{D_z}\rho_z z v_z dz\Big) - \int_{D_x}\rho_x v_x^2\, dx$$
$$- \int_{D_y}\rho_y u_y^2 dy - \int_{D_z}\rho_z (u_z^2+v_z^2)dz + \frac{1}{A}\Big(\int_{D_x}\rho_x x w_x dx + \int_{D_z}\rho_z z u_z dz\Big)^2$$
$$+ \frac{1}{B}\Big(\int_{D_y}\rho_y y w_y dy + \int_{D_z}\rho_z z v_z dz\Big)^2 + (m-m_x-m_z)y_c^2 + (m-m_y-m_z)x_c^2\Big] \qquad (55)$$

From Eq.(55) we note that the terms involving x_c^2 and y_c^2 are always positive so that, defining a new testing functional $\kappa_3|_E$ which is obtained from $\kappa_2|_E$ by setting $x_c = y_c = 0$, we can conclude that

$$\kappa_3|_E \leq \kappa_2|_E \qquad (56)$$

It is clear that the case where the motion of the mass center in the x and y direction is zero is the most restrictive case and the satisfaction of stability criteria obtained by ignoring this motion ensures stability for cases with arbitrary center of mass motion. In view of this, in the sequal we shall ignore the motion of the mass center. We note at this point that $\kappa_3|_E$ is still in a form not easily tested for sign definiteness. We shall now consider two methods for circumventing this problem, namely, the modal analysis and the method of integral coordinates.

Normal Mode Stability Analysis

a. General derivations.

We recall that the elastic displacements u_y, u_z, --- , w_y are assumed to satisfy individual eigenvalue problems defined by differential equations (29) and boundary conditions (28). At this point, we consider the eigenvalue problem given by

$$\mathcal{L}_u[u] = \Lambda_u^2 M_u[u] \tag{57}$$

where $\mathcal{L}_u$ is a linear homogeneous self-adjoint differential operator and M_u is merely the function ρ. Under these conditions, the function $u(P,t)$ may be represented by a superposition of space-dependent normal modes $\phi_i(P)$ multiplying corresponding time-dependent coordinates $\eta_i(t)$

$$u(P,t) = \sum_{i=1}^{n} \phi_i(P)\eta_i(t) \tag{58}$$

where P represents the point x, y, z. Furthermore, the eigenfunctions $\phi_i(P)$ are orthogonal and, if they are normalized such that

$$\int_D \rho(P)\phi_i(P)\phi_j(P)\ dD(P) = \delta_{ij} \tag{59}$$

it follows that

$$\int_D \phi_i(P)\ \mathcal{L}_u[\phi_j(P)]dD(P) = \Lambda_{ui}^2\ \delta_{ij} \tag{60}$$

where δ_{ij} represents the Kronecker delta.

We shall use these results later to eliminate the spatial dependence in $\kappa_3|_E$. In this section we shall consider a testing functional slightly different from $\kappa_3|_E$. Recalling Eq.(31) we note that V_{EL} was replaced by a lower bound using Rayleigh's quotient. In using modal analysis this yields no particular advantage and hence we consider the testing functional $\kappa_4|_E$ defined by

$$\kappa_4|_E = \frac{1}{2}\,\beta_0^2\{\ell\}^T[K]\{\ell\}\Big|_E + V_{EL} \tag{61}$$

which represents the original dynamic potential evaluated at equilibrium. We note again that in the first term of Eq.(61) the motion of the mass center is ignored. In analogy with previous reasoning, if $\kappa_4|_E$ is positive definite the equilibrium point is asymptotically stable.

We shall now consider the form of the elastic potential energy. To this end, we must take into account the effect of the centrifugal forces. Because the satellite has significant spin about axis z, whereas the angular velocities about axes x and y are relatively small, the centrifugal forces acting over the domains D_x, D_y, and D_z are all different. First we wish to distinguish between in-plane and out-of-plane vibrations of the rods associated with domains D_x and D_y. Moreover, we must distinguish between axial and transverse components of the centrifugal forces. It is not difficult to show that domains

D_x and D_y are subjected to the axial component of the centrifugal force alone for the out-of-plane vibration and to both the axial and transverse components for the in-plane vibration. On the other hand, domain D_z is subjected to the transverse component alone. The transverse components are accounted for in that part of the kinetic energy not involving velocities, so that only the axial centrifugal forces must be included in the elastic potential energy. Hence, the potential energy can be written in the form

$$V_{EL} = V_{ELx} + V_{ELy} + V_{ELz} \tag{62}$$

where

$$\begin{aligned} V_{ELx} &= \frac{1}{2}\int_{D_x} [EI_{v_x}\left(\frac{\partial^2 v_x}{\partial x^2}\right)^2 + EI_{w_x}\left(\frac{\partial^2 w_x}{\partial x^2}\right)^2]\, dx \\ &+ \frac{1}{2}\int_{D_x} P_x[\left(\frac{\partial v_x}{\partial x}\right)^2 + \left(\frac{\partial w_x}{\partial x}\right)^2]\, dx \\ V_{ELy} &= \frac{1}{2}\int_{D_y} [EI_{u_y}\left(\frac{\partial^2 u_y}{\partial y^2}\right)^2 + EI_{w_y}\left(\frac{\partial^2 w_y}{\partial y^2}\right)^2]\, dy \\ &+ \frac{1}{2}\int_{D_y} P_y[\left(\frac{\partial u_y}{\partial y}\right)^2 + \left(\frac{\partial w_y}{\partial y}\right)^2]\, dy \\ V_{ELz} &= \frac{1}{2}\int_{D_z} [EI_{u_z}\left(\frac{\partial^2 u_z}{\partial z^2}\right)^2 + EI_{v_z}\left(\frac{\partial^2 v_z}{\partial z^2}\right)^2]\, dz \end{aligned} \tag{63}$$

where P_x and P_y represent the axial centrifugal forces present (see, for example, Ref. 77, p. 443).

The elastic potential energy can be written in a more convenient form. To this end, we recall that the boundary conditions for the clamped-free rod corresponding to the domain $h_x < x < h_x + \ell_x$ are

$$v_x(x,t) = \frac{\partial v_x(x,t)}{\partial x} = 0 \text{ at } x = h_x \;,$$

$$EI_{v_x} \frac{\partial^2 v_x(x,t)}{\partial x^2} = \frac{\partial}{\partial x}[EI_{v_x} \frac{\partial^2 v_x(x,t)}{\partial x^2}] = 0 \text{ at } x = h_x + \ell_x \tag{64}$$

Similar boundary conditions can be written for the remaining rods. In view of this, integrating Eqs.(63) by parts and inserting the result in (62), we obtain

$$\begin{aligned} V_{EL} = \frac{1}{2}\{\int_{D_x} & [v_x \frac{\partial^2}{\partial x^2} (EI_{v_x} \frac{\partial^2 v_x}{\partial x^2}) + w_x \frac{\partial^2}{\partial x^2} (EI_{w_x} \frac{\partial^2 w_x}{\partial x^2})]\, dx \\ & - \int_{D_x} [v_x \frac{\partial}{\partial x} (P_x \frac{\partial v_x}{\partial x}) + w_x \frac{\partial}{\partial x} (P_x \frac{\partial w_x}{\partial x}]\, dx \\ & + \int_{D_y} [u_y \frac{\partial^2}{\partial y^2} (EI_{u_y} \frac{\partial^2 u_y}{\partial y^2}) + w_y \frac{\partial^2}{\partial y^2} (EI_{w_y} \frac{\partial^2 w_y}{\partial y^2})]\, dy \\ & - \int_{D_y} [u_y \frac{\partial}{\partial y} (P_y \frac{\partial u_y}{\partial y}) + w_y \frac{\partial}{\partial y} (P_y \frac{\partial w_y}{\partial y})]\, dy \\ & + \int_{D_z} [u_z \frac{\partial^2}{\partial z^2} (EI_{u_z} \frac{\partial^2 u_z}{\partial z^2}) + v_z \frac{\partial^2}{\partial z^2} (EI_{v_z} \frac{\partial^2 v_z}{\partial z^2})]\, dz\} \end{aligned} \tag{65}$$

The complete expression of $\kappa_4|_E$ is obtained by inserting expression (65) into (61). In accordance with Eq.(58), we represent the elastic displacements by the following series

$$\left.\begin{aligned} v_x &= \sum_{i=1}^{o_x} \phi_{xoi}(x)V_{xoi}(t) + \sum_{i=1}^{e_x} \phi_{xei}(x)V_{xei}(t) \\ w_x &= \sum_{i=1}^{o_x} \psi_{xoi}(x)W_{xoi}(t) + \sum_{i=1}^{e_x} \psi_{xei}(x)W_{xei}(t) \end{aligned}\right\} \text{ over } D_x \tag{66a}$$

$$\left.\begin{aligned} u_y &= \sum_{i=1}^{o_y} \phi_{yoi}(y)U_{yoi}(t) + \sum_{i=1}^{e_y} \phi_{yei}(y)U_{yei}(t) \\ w_y &= \sum_{i=1}^{o_y} \psi_{yoi}(y)W_{yoi}(t) + \sum_{i=1}^{e_y} \psi_{yei}(y)W_{yei}(t) \end{aligned}\right\} \text{ over } D_y \tag{66b}$$

$$\left.\begin{aligned} u_z &= \sum_{i=1}^{o_z} \phi_{zoi}(z)U_{zoi}(t) + \sum_{i=1}^{e_z} \phi_{zei}(z)U_{zei}(t) \\ v_z &= \sum_{i=1}^{o_z} \psi_{zoi}(z)V_{zoi}(t) + \sum_{i=1}^{e_z} \psi_{zei}(z)V_{zei}(t) \end{aligned}\right\} \text{ over } D_z \tag{66c}$$

where o_x, e_x, o_y, e_y, o_z, e_z are constant integers, ϕ_{xoi}, ϕ_{xei}, ψ_{xoi}, ---, ψ_{zei} are eigenfunctions associated with the elastic rods, and V_{xoi}, V_{xei}, W_{xoi}, ---, V_{zei} are corresponding generalized coordinates, in which the letters o and e designate odd and even modes of deformation, respectively. The functions ϕ_{xoi}, ϕ_{xei}, ψ_{xoi}, ---, ψ_{xei} satisfy the relations

$$\begin{aligned} \phi_{xoi}(x) &= -\phi_{xoi}(-x) = \phi_{xei}(x) = \phi_{xei}(-x) \\ \psi_{xoi}(x) &= -\psi_{xoi}(-x) = \psi_{xei}(x) = \psi_{xei}(-x) \end{aligned} \tag{67a}$$

$$\phi_{yoi}(y) = -\phi_{yoi}(-y) = \phi_{yei}(y) = \phi_{yei}(-y)$$
$$\psi_{yoi}(y) = -\psi_{yoi}(-y) = \psi_{yei}(y) = \psi_{yei}(-y) \tag{67b}$$

$$\phi_{zoi}(z) = -\phi_{zoi}(-z) = \phi_{zei}(z) = \phi_{zei}(-z)$$
$$\psi_{zoi}(z) = -\psi_{zoi}(-z) = \psi_{zei}(z) = \psi_{zei}(-z) \tag{67c}$$

Consistent with our previous discussion of the nature of the centrifugal forces, we recognize that the eigenfunctions entering into expressions (66) are defined by two distinct types of eigenvalue problems, namely, one type for the vibration of the radial rods associated with domains D_x and D_y and another type for the axial rods associated with domain D_z. For the radial rods, a typical eigenfunction, say ψ_{xoi}, must satisfy the differential equation

$$\frac{d^2}{dx^2}\left(EI_{w_x}\frac{d^2\psi_{xoi}}{dx^2}\right) - \frac{d}{dx}\left(P_x\frac{d\psi_{xoi}}{dx}\right) = \Lambda^2_{wxi}\,\rho_x\psi_{xoi},\ i=1,2,--- \tag{68}$$

over the domain $h_x < x < h_x + \ell_x$, where ψ_{xoi} is subject to the boundary conditions

$$\psi_{xoi}(h_x) = \left.\frac{d\psi_{xoi}}{dx}\right|_{x=h_x} = 0$$
$$i = 1,2,\ --- \tag{69}$$
$$EI_{w_x}\left.\frac{d^2\psi_{xoi}}{dx^2}\right|_{x=h_x+\ell_x} = \left[\frac{d}{dx}\left(EI_{w_x}\frac{d^2\psi_{xoi}}{dx^2}\right) - P_x\frac{d\psi_{xoi}}{dx}\right]\Bigg|_{x=h_x+\ell_x} = 0$$

The quantities Λ_{wxi}^2 (i=1,2,---) are the associated eigenvalues. Similar eigenvalue problems can be defined for ϕ_{xoi}, ϕ_{xei}, ψ_{xei}, ϕ_{yoi}, ϕ_{yei}, ψ_{yoi}, and ψ_{yei}. The solution of the eigenvalue problem defined by Eqs.(68) and (69) is discussed in Ref. 17 (see Sec. 10-4).

The axial rods are not subject to axial forces, so that a typical eigenvalue problem, say for ϕ_{zoi}, is defined by the differential equation

$$\frac{d^2}{dz^2}\left(EI_{u_z}\frac{d^2\phi_{zoi}}{dz^2}\right) = \Lambda_{uzi}^2\,\rho_z\,\phi_{zoi}\ ,\ i = 1,2,\ --- \tag{70}$$

which must be satisfied over the domain $h_z < z < h_z + \ell_z$, where the function ϕ_{zoi} is subject to boundary conditions of the form (69) with $P_x = 0$. Similar eigenvalue problems can be defined for ϕ_{zei}, ψ_{zoi}, and ψ_{zei}. If the rod is uniform, the solution of the eigenvalue problem can be taken directly from Ref. 17 (Sec. 5-10).

For uniform or nonuniform rods the solution of the eigenvalue problem (68) can be obtained by one of the approximate methods described in Ref. 17 (Ch. 6), and the same can be said about the eigenvalue problem (70) if the rod is nonuniform. In the sequel we shall regard all the eigenfunctions and associated eigenvalues as known.

The eigenfunctions possess the orthogonality property. Moreover, they can be normalized, so that

$$\int_{D_x} \rho_x \psi_{xoi}(x)\psi_{xoj}(x)\, dx = 2\delta_{ij}$$

$$\int_{D_x} \rho_x \psi_{xei}(x)\psi_{xej}(x)\, dx = 2\delta_{ij} \qquad i,j = 1,2, --- \tag{71}$$

$$\int_{D_x} \rho_x \psi_{xoi}(x)\psi_{xej}(x)\, dx = 0$$

where δ_{ij} is the Kronecker delta. Similar expressions can be written for the remaining eigenfunctions.

In view of the above, a typical term in expression (65) becomes

$$\int_{D_x} w_x [\frac{\partial^2}{\partial x^2}(EI_{w_x} \frac{\partial^2 w_x}{\partial x^2}) - \frac{\partial}{\partial x}(P_x \frac{\partial w_x}{\partial x})]\, dx = \int_{D_x} (\sum_{i=1}^{o_x} \psi_{xoi} w_{xoi}$$

$$+ \sum_{i=1}^{e_x} \psi_{xei} w_{xei}) \times \{ \sum_{j=1}^{o_x} w_{xoi} [\frac{d^2}{dx^2}(EI_{w_x} \frac{d^2\psi_{xoj}}{dx^2}) - \frac{d}{dx}(P_x \frac{d\psi_{xoj}}{dx})]$$

$$+ \sum_{j=1}^{e_x} w_{xei} [\frac{d^2}{dx^2} (EI_{w_x} \frac{d^2\psi_{xej}}{dx^2}) - \frac{d}{dx} (P_x \frac{d\psi_{xej}}{dx})] \}\, dx$$

$$= 2 (\sum_{i=1}^{o_x} \Lambda_{wxi}^2 W_{xoi}^2 + \sum_{i=1}^{e_x} \Lambda_{wxi}^2 W_{xei}^2) \tag{72}$$

Hence, the potential energy V_{EL} can be regarded as a function of the generalized coordinates V_{xoi}, V_{xei}, W_{xoi}, etc. It follows that $\kappa_4|_E$, Eq.(61), is a quadratic form in the $2(1 + o_x + e_x + o_y + --- + e_z)$ variables $\theta_1, \theta_2, W_{yoi}, W_{yei}$, ---. For stability, $\kappa_4|_E$ must be positive definite in these variables.

Furthermore, by using even and odd modes to represent the elastic displacements no coupling between even and odd modes occurs. Hence, $\kappa_4|_E$ may be represented as the sum of two quadratic forms $\kappa_{4o}|_E$ and $\kappa_{4e}|_E$, where $\kappa_{4e}|_E$ involves only even modes and $\kappa_{4o}|_E$ involves odd modes and the rigid body motion. Therefore, we have

$$\kappa_4|_E = \kappa_{4o}|_E + \kappa_{4e}|_E \tag{73}$$

where

$$\begin{aligned}\kappa_{4e}|_E = &\sum_{i=1}^{e_x}[(\Lambda_{vxi}^2 - \Omega_s^2)V_{xei}^2 + \Lambda_{wxi}^2 W_{xei}^2] \\ &+ \sum_{i=1}^{e_y}[(\Lambda_{uyi}^2 - \Omega_s^2)W_{yei}^2 + \Lambda_{wyi}^2 W_{yei}^2] \\ &+ \sum_{i=1}^{e_z}[(\Lambda_{vzi}^2 - \Omega_s^2)V_{zei}^2 + (\Lambda_{uzi}^2 - \Omega_s^2)U_{zei}^2]\end{aligned} \tag{74a}$$

and

$$\begin{aligned}\kappa_{4o}|_E = &\frac{1}{2}\,\Omega_s^2[\frac{C}{B}\,(C-B)\,\theta_1^2 + \frac{C}{A}\,(C-A)\,\theta_2^2 \\ &+ 4\,\theta_1\,\frac{C}{B}(\sum_{i=1}^{o_y} J_{wyi}W_{yoi} + \sum_{i=1}^{o_z} J_{vzi}V_{zoi}) \\ &- 4\,\theta_2\,\frac{C}{A}(\sum_{i=1}^{o_x} J_{wxi}W_{xoi} + \sum_{i=1}^{o_z} J_{uzi}U_{zoi})] \\ &+ \sum_{i=1}^{o_x}\sum_{j=1}^{o_x}(\Lambda_{wxi}^2\,\delta_{ij} + \frac{2}{A}\,\Omega_s^2\,J_{wxi}J_{wxj})W_{xoi}W_{xoj}\end{aligned}$$

$$+\sum_{i=1}^{o_y}\sum_{j=1}^{o_y}(\Lambda_{wyi}^2\delta_{ij}+\frac{2}{B}\,\Omega_s^2\,J_{wyi}J_{wyj})W_{yoi}W_{yoj}$$

$$+\sum_{i=1}^{o_z}\sum_{j=1}^{o_z}[(\Lambda_{uzi}^2-\Omega_s^2)\delta_{ij}+\frac{2}{A}\,\Omega_s^2\,J_{uzi}J_{uzj}]U_{zoi}U_{zoj}$$

$$+\sum_{i=1}^{o_z}\sum_{j=1}^{o_z}[(\Lambda_{vzi}^2-\Omega_s^2)\delta_{ij}+\frac{2}{B}\,\Omega_s^2\,J_{vzi}J_{vzj}]V_{zoi}V_{zoj}$$

$$+\frac{4}{A}\,\Omega_s^2\sum_{i=1}^{o_x}\sum_{j=1}^{o_z}J_{wxi}J_{uzj}W_{xoi}U_{zoj}+\frac{4}{B}\Omega_s^2\sum_{i=1}^{o_y}\sum_{j=1}^{o_z}J_{wyi}J_{vzj}W_{yoi}V_{zoj}$$

$$+\sum_{i=1}^{o_x}(\Lambda_{vxi}^2-\Omega_s^2)V_{xoi}^2+\sum_{i=1}^{o_y}(\Lambda_{uyi}^2-\Omega_s^2)W_{yoi}^2 \tag{74b}$$

in which

$$J_{vxi}=\int_{h_x}^{h_x+\ell_x}\rho_x x\phi_{xoi}\,dx\,,\quad J_{wxi}=\int_{h_x}^{h_x+\ell_x}\rho_x x\psi_{xoi}\,dx$$

$$J_{uyi}=\int_{h_y}^{h_y+\ell_y}\rho_y y\phi_{yoi}\,dy\,,\quad J_{wyi}=\int_{h_y}^{h_y+\ell_y}\rho_y y\psi_{yoi}\,dy \tag{75}$$

$$J_{uzi}=\int_{h_z}^{h_z+\ell_z}\rho_z z\phi_{zoi}\,dz\,,\quad J_{vzi}=\int_{h_z}^{h_z+\ell_z}\rho_z z\psi_{zoi}\,dz$$

We recall that $\kappa_4|_E$ must be positive definite for the equilibrium point to be asymptotically stable. Since Eq.(73) $\kappa_4|_E$ can be written as the sum of two parts, $\kappa_{4o}|_E$ and $\kappa_{4e}|_E$, it follows that for $\kappa_4|_E$ to be positive definite it is necessary that both $\kappa_{4o}|_E$ and $\kappa_{4e}|_E$ be positive definite.

The expressions for $\kappa_{4e}|_E$ and $\kappa_{4o}|_E$ can be written in the general form

$$\kappa_{4o}|_E = \frac{1}{2} \sum_{i=1}^{n_o} \sum_{j=1}^{n_o} \alpha_{oij}\, q_{oi}\, q_{oj}$$
$$\kappa_{4e}|_E = \frac{1}{2} \sum_{i=1}^{n_e} \sum_{j=1}^{n_e} \alpha_{eij}\, q_{ei}\, q_{ej} \tag{76}$$

where q_{oi} and q_{ei} are generalized coordinates and n_o and n_e are integers denoting the number of coordinates q_{ei} and q_{oi} considered. The integers n_o and n_e depend on the number of modes assumed and, hence, on the integers e_x, e_y, e_z, o_x, o_y, o_z. The quantities α_{eij} and α_{oij} represent constant coefficients. According to Sylvester's criterion $\kappa_{4o}|_E$ and $\kappa_{4e}|_E$ are positive definite if the conditions

$$|\alpha_{oij}| > 0 \;,\; |\alpha_{eij}| > 0 \quad i,j=1,2,---,k;k=1,2,---,n \tag{77}$$

are satisfied, where $|\alpha_{oij}|$ and $|\alpha_{eij}|$ are the principal minor determinants associated with matrices $[\alpha_o]$ and $[\alpha_e]$ of the coefficients. Matrices $[\alpha_o]$ and $[\alpha_e]$ are referred to as Hessian matrices.

Using Eq.(74a), we may write the Hessian matrix $[\alpha_e]$ in the diagonal form

$$[\alpha_e] = \lceil a_i \rfloor \tag{78}$$

where the order of the matrix is

$$n_e = 2e_x + 2e_y + 2e_z \tag{79}$$

For a diagonal matrix conditions (77) merely imply that each diagonal element must be positive. Hence, $\kappa_{4e}\big|_E$ is positive definite if

$$a_i > 0 \qquad i = 1, ---, n_e \tag{80}$$

where the constants a_i are defined by

$$a_i = \begin{cases} \Lambda^2_{vxi}-\Omega^2_s & i=1,---,e_x \\ \Lambda^2_{wxj} & j=i-e_x \ ; \ i=1+e_x,---,2e_x \\ \Lambda^2_{uyj}-\Omega^2_s & j=i-2e_x \ ; \ i=1+2e_x,---,2e_x+e_y \\ \Lambda^2_{wyj} & j=i-2e_x-e_y;i=1+2e_x+e_y,---,2e_x+2e_y \\ \Lambda^2_{uzj}-\Omega^2_s & j=i-2e_x-2e_y;i=1+2e_x+2e_y,---,2e_x+2e_y+e_z \\ \Lambda^2_{vzj}-\Omega^2_s & j=1-2e_x-2e_y-e_z;i=1+2e_x+2e_y+e_z,---,n_e \end{cases} \tag{81}$$

Considering the above definitions we see that the conditions given in (80) are met if

$$\Lambda^2_{vxi} > \Omega^2_s \tag{82a}$$

$$i = 1, ---, e_x$$

$$\Lambda^2_{wxi} > 0 \tag{82b}$$

$$\Lambda^2_{uyi} > \Omega^2_s \tag{82c}$$

$$i = 1, ---, e_y$$

$$\Lambda^2_{wyi} > 0 \tag{82d}$$

$$\Lambda^2_{uzi} > \Omega^2_s \qquad (82e)$$

$$i = 1, ---, e_z$$

$$\Lambda^2_{vzi} > \Omega^2_s \qquad (82f)$$

By inspection it is obvious that inequalities (82b) and (82d) are met. Furthermore, we recall that Λ_{vx1}, Λ_{vz1}, Λ_{uz1} and Λ_{uy1} are the lowest natural frequencies associated with these vibrations, so that inequalities (82) are satisfied if

$$\Lambda_{vx1} > \Omega_s \quad , \quad \Lambda_{uy1} > \Omega_s \qquad (83a)$$

$$\Lambda_{vz1} > \Omega_s \quad , \quad \Lambda_{uz1} > \Omega_s \qquad (83b)$$

where Λ_{vx1} and Λ_{uy1} are defined by eigenvalue problems similar to those given by (68) and (69), whereas Λ_{vz1} and Λ_{uz1} are defined by eigenvalue problems similar to those given by (70) and (69). Considering Eq.(68), it is possible to show that conditions (83a) are always met (see Appendix A). Hence, the testing function $\kappa_{4e}|_E$ is positive definite if conditions (83b) are satisfied.

We shall now consider the function $\kappa_{4o}|_E$ and note that it can be written as the sum of four independent quadratic forms

$$\kappa_{4o}|_E = \kappa_{4o1}|_E + \kappa_{4o2}|_E + \kappa_{4o3}|_E + \kappa_{4o4}|_E \qquad (84)$$

where

$$\kappa_{4o1}|_E = \sum_{i=1}^{o_x} (\Lambda^2_{vxi} - \Omega^2_s) V^2_{xoi} \qquad (85a)$$

$$\kappa_{4o2}\big|_E = \sum_{i=1}^{o_y} (\Lambda_{uyi}^2 - \Omega_s^2) W_{yoi}^2 \tag{85b}$$

$$\kappa_{4o3}\big|_E = \frac{1}{2}\,\Omega_s^2 [\tfrac{C}{B}(C-B)\theta_1^2 + 4\theta_1 \tfrac{C}{B} (\sum_{i=1}^{o_y} J_{wyi}\, W_{yoi}$$

$$+ \sum_{i=1}^{o_z} J_{vzi} V_{zoi})] + \sum_{i=1}^{o_y} \sum_{j=1}^{o_y} (\Lambda_{wyi}^2 \delta_{ij} + \tfrac{2}{B}\,\Omega_s^2\, J_{wyi} J_{wyj}) W_{yoi} W_{yoj}$$

$$+ \sum_{i=1}^{o_z} \sum_{j=1}^{o_z} [(\Lambda_{vzi}^2 - \Omega_s^2)\delta_{ij} + \tfrac{2}{B}\,\Omega_s^2\, J_{vzi} J_{vzj}] V_{zoi} V_{zoj}$$

$$+ \frac{4}{B}\,\Omega_s^2 \sum_{i=1}^{o_y} \sum_{j=1}^{o_z} J_{wyi}\, J_{vzj}\, W_{yoi}\, V_{zoj} \tag{85c}$$

$$\kappa_{4o4}\big|_E = \frac{1}{2}\,\Omega_s^2 [\tfrac{C}{A}\,(C-A)\theta_2^2 - 4\theta_2 \tfrac{C}{A} (\sum_{i=1}^{o_x} J_{wxi}\, W_{xoi}$$

$$+ \sum_{i=1}^{o_z} J_{uzi} U_{zoi})] + \sum_{i=1}^{o_x} \sum_{j=1}^{o_x} (\Lambda_{wxi}^2 \delta_{ij} + \tfrac{2}{A}\,\Omega_s^2\, J_{wxi} J_{wxj}) W_{xoi} W_{xoj}$$

$$+ \sum_{i=1}^{o_z} \sum_{j=1}^{o_z} [(\Lambda_{uzi}^2 - \Omega_s^2)\delta_{ij} + \tfrac{2}{A}\,\Omega_s^2\, J_{uzi} J_{uzj}] U_{zoi} U_{zoj}$$

$$+ \frac{4}{A}\,\Omega_s^2 \sum_{i=1}^{o_x} \sum_{j=1}^{o_z} J_{wxi}\, J_{uzj}\, W_{xoi}\, U_{zoj} \tag{85d}$$

By inspection, we see that $\kappa_{4o1}\big|_E$ and $\kappa_{4o2}\big|_E$ are positive definite if $\Lambda_{vx1} > \Omega_s$ and $\Lambda_{uy1} > \Omega_s$. As discussed previously, this is always true, so that $\kappa_{4o}\big|_E$ is positive definite if

$\kappa_{4o3}|_E$ and $\kappa_{4o4}|_E$ are both positive definite.

Let us consider first the function $\kappa_{4o4}|_E$. For conveni-ence, we define the following substitutions

$$\eta_0 = - \theta_2 \frac{C\Omega_s}{(2A)^{1/2}} \tag{86a}$$

$$\eta_i = \begin{cases} U_{zoi}(2/A)^{1/2}\ \Omega_s J_{uzi} & i=1,---,o_z \\ W_{xoj}(2/A)^{1/2}\ \Omega_s J_{wxj} & j=i-o_z \\ & i=1+o_z,---,o_x+o_z \end{cases} \tag{86b}$$

In terms of these new variables, $\kappa_{4o4}|_E$ can be written as

$$\kappa_{4o4}|_E = \sum_{i=0}^{o_x+o_z} \sum_{j=0}^{o_x+o_z} (1 + b_i\delta_{ij})\eta_i\eta_j \tag{87}$$

where

$$b_0 = - \frac{A}{C} \tag{88a}$$

$$b_i = \begin{cases} \dfrac{A(\Lambda^2_{uzi}/\Omega^2_s - 1)}{2\ J^2_{uzi}} & i = 1,\ ---,\ o_z \\ \dfrac{A(\Lambda_{wxj}/\Omega_s)^2}{2\ J^2_{wxj}} & j = i - o_z \\ & i = 1+o_z,---,o_x+o_z \end{cases} \tag{88b}$$

Hence, the Hessian matrix associated with $\kappa_{4o4}|_E$ is given by

$$[\mathcal{H}_{4o4}]_E = [1 + b_i\delta_{ij}] \tag{89}$$

where i takes values from 0 to $o_x + o_z$. Denoting the principal minor determinants of $[\mathcal{H}_{4o4}]_E$ by $\Delta_i (i = 0,---, o_x+o_z)$ the stability conditions are

$$\Delta_i > 0 \qquad i = 0,1,---,o_x+o_z \tag{90}$$

where

$$\Delta_0 = b_0(1 + \frac{1}{b_0})$$

$$\Delta_1 = b_0 b_1 (1 + \frac{1}{b_0} + \frac{1}{b_1}) \tag{91}$$

$$\Delta_p = \prod_{i=0}^{p} b_i (1 + \sum_{i=0}^{p} b_i^{-1}) \qquad p = 0,---,o_x+o_z$$

Considering Eqs.(88) and inequalities (83b), we see that if $\kappa_{4e}|_E$ is positive definite, then the $b_i (i = 1,---, o_x+o_z)$ are all positive and b_0 is negative. Under these circumstances the requirements $\Delta_i > 0$, $(i = 0,1,---,o_x+o_z)$ can be written as

$$\sum_{i=0}^{p} \frac{1}{b_i} + 1 < 0 \;, \; p = 0,---, o_x+o_z \tag{92}$$

In terms of the system parameters (92) yields the following stability criteria

$$C > A \tag{93a}$$

$$(\frac{\Lambda_{wx1}}{\Omega_s})^2 > \frac{2\, J_{wx1}^2}{C-A-2 \sum_{i=2}^{o_x} J_{wxi}^2/(\Lambda_{wxi}/\Omega_s)^2} \tag{93b}$$

$$\left(\frac{\Lambda_{uz1}}{\Omega_s}\right)^2 > 1 + \frac{2\,J_{uz1}^2}{C-A-2\sum_{i=2}^{o_z} J_{uzi}^2/(\Lambda_{uzi}^2/\Omega_s^2-1)-2\sum_{i=1}^{o_x} J_{wxi}^2/(\Lambda_{wxi}/\Omega_s)^2} \tag{93c}$$

Following a procedure similar to that used above, the requirements for $\kappa_{4o3}\big|_E$ to be positive definite are

$$C > B \tag{94a}$$

$$\left(\frac{\Lambda_{wy1}}{\Omega_s}\right)^2 > \frac{2\,J_{wy1}^2}{C-B-2\sum_{i=2}^{o_y} J_{wyi}^2/(\Lambda_{wyi}/\Omega_s)^2} \tag{94b}$$

$$\left(\frac{\Lambda_{vz1}}{\Omega_s}\right)^2 > 1 + \frac{2\,J_{vz1}^2}{C-B-2\sum_{i=2}^{o_z} J_{vzi}^2/(\Lambda_{vzi}^2/\Omega_s^2-1)-2\sum_{i=1}^{o_y} J_{wyi}^2/(\Lambda_{wyi}/\Omega_s)^2} \tag{94c}$$

Recalling Eq.(73), we see that $\kappa_4\big|_E$ is positive definite if conditions (83b), (93) and (94) are satisfied. However, we notice that conditions (83b) are contained in (93c) and (94c), so that Eqs.(93) and (94) present a complete stability picture. It should be noted, that while in general conditions (93) and (94) apply to a satellite with three pairs of symmetric flexible rods, they can also be applied to a satellite containing any smaller number of symmetric pairs of these rods. To this end, we note from Eqs.(75) that if the length of any pair of rods becomes zero the corresponding J_{vxi}, J_{wxi},---, or J_{vzi} becomes zero and the series representing that particular pair of rods is identically zero. Finally, it should be noted that inequalities (93) and (94) have identical forms, so that in the stability criteria established considering inequalities (93) we can substitute for the parameters A, Λ_{uzi}, Λ_{wxi}, J_{uzi} and J_{wxi} the parameters B, Λ_{vzi}, Λ_{wyi}, J_{vzi} and J_{wyi} to obtain stability criteria corresponding to inequalities (94). In the sequal we shall be concerned only with inequalities (93) and results obtained using these inequalities will be applied to inequalities (94) using the substitutions defined above.

A check as to whether inequalities (93) are satisfied will be performed numerically. For convenience, we shall write inequalities (93b) and (93c) in the slightly different form

$$\frac{\Omega_s}{\Lambda_{wx1}} < \left[\frac{(C-A)/A_0 - 2\sum_{i=2}^{o_x}(J_{wxi}^2/A_0)/(\Lambda_{wxi}/\Omega_s)^2}{2\,J_{wx1}^2/A_0}\right]^{1/2} \tag{95a}$$

$$\frac{\Omega_s}{\Lambda_{uz1}} < \left[1 + \frac{2\,J_{uz1}^2/A_0}{(C-A)/A_0 - R}\right]^{-1/2} \tag{95b}$$

in which $(C-A)/A_0 = C_0/A_0 - 1 + R_{AX} - R_{AZ}$ and the parameter R is given by

$$R = 2\sum_{i=2}^{o_z}(J_{uzi}^2/A_0)/(\Lambda_{uzi}^2/\Omega_s^2 - 1) + 2\sum_{i=1}^{o_x}(J_{wxi}^2/A_0)/(\Lambda_{wxi}/\Omega_s)^2 \tag{96}$$

where C_0 and A_0 represent moments of inertia of the rigid part of the satellite about the axes z and x, respectively. The quantities R_{AX} and R_{AZ} represent the ratios

$$R_{AX} = \frac{2}{A_0} \int_{h_x}^{h_x+\ell_x} \rho_x x^2 \, dx \quad , \quad R_{AZ} = \frac{2}{A_0} \int_{h_z}^{h_z+\ell_z} \rho_z z^2 \, dz \tag{97}$$

At this point, a few comments about the nature of the stability criteria resulting from inequalities (93a) and (95) are in order. We note from (93a) that for stability spin should be imparted about the axis of maximum moment of inertia. Inequalities (95) indicate that the frequency ratios Ω_s/Λ_{wx1} and Ω_s/Λ_{uz1} are determined by the system parameters and, in particular, that Ω_s/Λ_{uz1} must not merely be less than unity as predicted by (83b) but its value must be according to (95b).

b. <u>Numerical solution</u>

If we let the thin elastic rods be uniform, the solution of Eq.(70), subject to boundary conditions (69) is

$$\phi_{zoi} = \frac{[\sin\beta_i\ell_z - \sinh\beta_i\ell_z][\sin\beta_i(z-h_z) - \sinh\beta_i(z-h_z)]}{(\rho_z\ell_z)^{1/2}\sin\beta_i\ell_z\sinh\beta_i\ell_z}$$

$$+ \frac{[\cos\beta_i\ell_z + \cosh\beta_i\Omega_z][\cos\beta_i(z-h_z) - \cosh\beta_i(z-h_z)]}{(\rho_z\ell_z)^{1/2}\sin\beta_i\ell_z\sinh\beta_i\ell_z} \tag{98}$$

in which $(\beta_i\ell_z)^4 = \Lambda_{uzi}^2 \, \rho_z\ell_z/EI_{uz}$, where $\beta_i\ell_z$ is determined by

$$\cos\beta_i\ell_z\cosh\beta_i\ell_z = -1 \tag{99}$$

Introducing Eq.(98) into Eqs.(75), we obtain

$$J_{uzi} = \int_{h_z}^{h_z+\ell_z} \rho_z z\phi_{zoi}\, dz = (m_z \ell_z^2)^{1/2} S_{zi} \tag{100}$$

where

$$S_{zi} = \frac{\sqrt{2}[(h_z/\ell_z)\beta_i\ell_z(\sin\beta_i\ell_z-\sinh\beta_i\ell_z)-(\cos\beta_i\ell_z+\cosh\beta_i\ell_z)]}{(\beta_i\ell_z)^2\sin\beta_i\ell_z\sinh\beta_i\ell_z} \tag{101}$$

and

$$J_{uzi}^2/A_0 = \frac{R_{AZ}S_{zi}^2}{\tau_z} \qquad i = 1, ---, o_z \tag{102}$$

where

$$\tau_z = [(h_z/\ell_z)^2 + (h_z/\ell_z) + 1/3] \tag{103}$$

We also note that the frequency ratios $(\Lambda_{uzi}/\Omega_s)^2$, $(i=2,---,o_z)$ are given by

$$(\Lambda_{uzi}/\Omega_s)^2 = (\beta_i\ell_z/\beta_1\ell_z)^4 \quad (\Lambda_{uz1}/\Omega_s)^2 \tag{104}$$

The terms involving the z rods in inequality (95b) are determined by using Eqs.(102) and (103).

The eigenvalue problem defined by Eq.(68) and subject to boundary conditions (69) must be solved using an approximate technique. We shall seek to set up the eigenvalue problem by means of Galerkin's method (see Ref. 17, Sec. 6.6). To this end, we assume a solution in the form of the series

$$\psi_{xon} = \sum_{i=1}^{n} a_i \, \hat{\psi}_{xoi} \tag{105}$$

where a_i are constant coefficients to be determined and $\hat{\psi}_{xoi}$ are comparison functions. We choose as comparison functions for the rotating rod the eigenfunctions of the cantilever rod obtained by setting $\Omega_s = 0$. We note that these eigenfunctions are given by Eq.(98) if ϕ_{zoi}, ℓ_z, z and ρ_z are replaced by $\hat{\psi}_{xoi}$, ℓ_x, x and ρ_x. Using Galerkin's method we obtain an algebraic eigenvalue problem defining n eigenvalues ${}^n\Lambda_{wxi}$ and the associated eigenvectors $\{a^{(i)}\}$

$$[k]\{a^{(i)}\} = {}^n\Lambda^2_{wxi} \, [m]\{a^{(i)}\} \tag{106}$$

where k_{ij} are obtained from

$$\begin{aligned} k_{ij} = k_{ji} &= \int_{h_x}^{h_x+\ell_x} EI_{wx} \frac{d^2\hat{\psi}_{xoi}}{dx^2} \frac{d^2\hat{\psi}_{xoj}}{dx^2} \, dx \\ &+ \frac{1}{4} m_x \, \Omega_s^2 (h_x+\ell_x)^2 \int_{h_x}^{h_x+\ell_x} [1 - \frac{x^2}{(h_x+\ell_x)^2}] \frac{d\hat{\psi}_{xoi}}{dx} \frac{d\hat{\psi}_{xoj}}{dx} \, dx \\ &= (\Lambda^2_{wxi})_{NR} \delta_{ij} + \frac{1}{4} m_x \Omega_s^2 (h_x+\ell_x)^2 \int_{h_x}^{h_x+\ell_x} [1 - \frac{x^2}{(h_x+\ell_x)^2}] \frac{d\hat{\psi}_{xoi}}{dx} \frac{d\hat{\psi}_{xoj}}{dx} \, dx \end{aligned} \tag{107}$$

in which $(\Lambda_{wxi})_{NR}$ represents the frequency for a nonrotating rod whose rotating natural frequency is Λ_{wxi}. The coefficients m_{ij} are obtained from

$$m_{ij} = m_{ji} = \int_{h_x}^{h_x+\ell_x} \rho_x \hat{\psi}_{xoi} \hat{\psi}_{xoj} \, dx = \delta_{ij} \tag{108}$$

The solution of Eq.(106) yields the coefficients $a_j^{(i)}$ (i,j=1,--- n) and the corresponding frequencies $^n\Lambda_{wxi}^2$. Recalling Eqs.(75), and using Eq.(105), the functions J_{wxi} are given by

$$\begin{aligned} J_{wxi} &= \int_{h_x}^{h_x+\ell_x} \rho_x x \sum_{j=1}^{n} a_j^{(i)} \hat{\psi}_{xoj} \, dx \\ &= (m_x \ell_x^2)^{1/2} \sum_{j=1}^{n} a_j^{(i)} R_{xj} = (m_x \ell_x^2)^{1/2} S_{xi} \end{aligned} \tag{109}$$

where

$$R_{xj} = \frac{\sqrt{2}[(h_x/\ell_x)\beta_j\ell_x(\sin\beta_j\ell_x - \sinh\beta_j\ell_x) - (\cos\beta_j\ell_x + \cosh\beta_j\ell_x)]}{(\beta_j\ell_x)^2 \sin\beta_j\ell_x \sinh\beta_j\ell_x} \tag{110}$$

and

$$J_{wxi}^2/A_0 = \frac{R_{AX} S_{xi}^2}{\tau_x} \tag{111}$$

in which

$$\tau_x = [(h_x/\ell_x)^2 + (h_x/\ell_x) + 1/3] \tag{112}$$

Equation (111) together with the solution of Eq.(106) permits the evaluation of the terms in inequalities (95) associated with the elastic displacement w_x.

At this point, a brief description of the numerical scheme appears in order. The values of Ω_s/Λ_{uz1}, R_{AZ}, h_z/ℓ_z, ℓ_x/ℓ_z, h_x/h_z, ρ_x/ρ_z, EI_{wx}/EI_{uz} and C_0/A_0 are fed into a computer program. The program inserts these values, together with those of $\beta_i\ell_z$ obtained by solving (99), into Eqs.(100) through (103) to evaluate J^2_{uzi}/A_0. Equations (107) and (108) are then used to define the eigenvalue problem (106). The eigenvalue problem is solved using IBM subroutines EIGEN and NROOT yielding the frequency ratios $(\Lambda_{wxi}/\Omega_s)^2$ and the constants $a^{(i)}_j$. Using Eqs.(109) through (111) the values of J^2_{wxi}/A_0 are also determined. With the values of J^2_{uzi}/A_0, J^2_{wxi}/A_0, $(\Lambda_{wxi}/\Omega_s)^2$ and $(\Lambda_{uzi}/\Omega_s)^2$ thus computed, the satisfaction of inequalities (95) can be checked. Results of these computations are presented later.

Method of Integral Coordinates

The stability analysis of the previous section has the disadvantage of leading to an involved numerical procedure. The effects of changes in various system parameters are not easily assessed. Moreover, in using the normal mode approach, the question as to the effect of series truncation on the accuracy of the results remains unanswered. For these reasons we shall seek closed-form stability criteria. We recall from Eq.(56) that for asymptotic stability the functional $\kappa_3|_E$ must be positive definite, where $\kappa_3|_E$ is given by

$$\kappa_3\Big|_E = \frac{1}{2}\{\Omega_s^2[\frac{C}{B}(C-B)\theta_1^2 + \frac{C}{A}(C-A)\theta_2^2 - 2\frac{C}{A}\theta_2(\int_{D_x}\rho_x x w_x dx$$

$$+ \int_{D_z}\rho_z z u_z dz) + 2\frac{C}{B}\theta_1(\int_{D_y}\rho_y y w_y dy + \int_{D_z}\rho_z z v_z dz) + \frac{1}{A}(\int_{D_x}\rho_x x w_x dx$$

$$+ \int_{D_z}\rho_z z u_z dz)^2 + \frac{1}{B}(\int_{D_y}\rho_y y w_y dy + \int_{D_z}\rho_z z v_z dz)^2] + \int_{D_x}\rho_x(\Lambda_{vx1}^2 - \Omega_s^2)v_x^2 dx$$

$$+ \int_{D_x}\rho_x\Lambda_{wx1}^2 w_x^2 dx + \int_{D_y}\rho_y(\Lambda_{uy1}^2 - \Omega_s^2)u_y^2 dy + \int_{D_y}\rho_y\Lambda_{wy1}^2 w_y^2 dy$$

$$+ \int_{D_z}\rho_z(\Lambda_{uz1}^2 - \Omega_s^2)u_z^2 dz + \int_{D_z}\rho_z(\Lambda_{vz1}^2 - \Omega_s^2)v_z^2 dz\} \tag{113}$$

Again we note that Eq. (113) is both a function and a functional and it may not be possible to determine its sign definiteness by standard techniques. However, by defining suitable new coordinates and using Schwarz's inequality for functions, it may be possible to circumvent this problem. To this end, we define the following integral coordinates

$$\bar{v}_x(t) = \int_{D_x}\rho_x x v_x(x,t)dx \ , \ \bar{w}_x(t) = \int_{D_x}\rho_x x w_x(x,t)dx$$

$$\bar{u}_y(t) = \int_{D_y}\rho_y y u_y(y,t)dy \ , \ \bar{w}_y(t) = \int_{D_y}\rho_y y w_y(y,t)dy \tag{114}$$

$$\bar{u}_z(t) = \int_{D_z}\rho_z z u_z(z,t)dz \ , \ \bar{v}_z(t) = \int_{D_z}\rho_z z v_z(z,t)dz$$

Using Schwarz's inequality, we have

$$\left(\int_{D_x} \rho_x x v_x dx\right)^2 \leq \int_{D_x} \rho_x x^2 dx \int_{D_x} \rho_x v_x^2 \, dx \tag{115}$$

Recalling the definition of $\bar{v}_x$, and solving for $\int_{D_x} \rho_x v_x^2 \, dx$, inequality (115) yields

$$\int_{D_x} \rho_x v_x^2 \, dx \geq \frac{\bar{v}_x^{\,2}}{A_0 R_{AX}} \tag{116}$$

Similarly

$$\int_{D_x} \rho_x w_x^2 \, dx \geq \frac{\bar{w}_x^{\,2}}{A_0 R_{AX}} \, , \quad \int_{D_y} \rho_y w_y^2 \, dy \geq \frac{\bar{w}_y^{\,2}}{B_0 R_{BY}}$$

$$\int_{D_y} \rho_y u_y^2 \, dy \geq \frac{\bar{u}_y^{\,2}}{B_0 R_{BY}} \, , \quad \int_{D_z} \rho_z u_z^2 \, dz \geq \frac{\bar{u}_z^{\,2}}{A_0 R_{AZ}} \tag{117}$$

$$\int_{D_z} \rho_z v_z^2 \, dz \geq \frac{\bar{v}_z^{\,2}}{B_0 R_{BZ}}$$

where the ratios R_{BY} and R_{BZ} are given by

$$R_{BY} = \frac{1}{B_0} \int_{D_y} \rho_y y^2 \, dy \, , \quad R_{BZ} = \frac{1}{B_0} \int_{D_z} \rho_z z^2 \, dz \tag{118}$$

and B_0 denotes the mass moment of inertia of the rigid part of the satellite about the y axis. Inserting Eqs.(117) into Eq.(113), noting that $\Lambda_{vx1} > \Omega_s$ and $\Lambda_{uy1} > \Omega_s$, and if, in addition, we assume $\Lambda_{uz1} > \Omega_s$ and $\Lambda_{vz1} > \Omega_s$ (which will later be shown to be the case), then we can define a new testing function $\kappa_5|_E$ given by

$$\kappa_5|_E = \frac{1}{2}\{\Omega_s^2[\frac{C}{B}(C-B)\theta_1^2 + \frac{C}{A}(C-A)\theta_2^2 - 2\frac{C}{A}(\bar{w}_x + \bar{u}_z)$$

$$+ 2\frac{C}{B}(\bar{w}_y + \bar{v}_z) + \frac{1}{A}(\bar{w}_x + \bar{u}_z)^2 + \frac{1}{B}(\bar{w}_y + \bar{v}_z)^2]$$

$$+ \frac{(\Lambda_{vx1}^2 - \Omega_s^2)}{A_0 R_{AX}}\bar{v}_x{}^2 + \frac{\Lambda_{wx1}^2}{A_0 R_{AX}}\bar{w}_x{}^2 + \frac{(\Lambda_{uy1}^2 - \Omega_s^2)}{B_0 R_{BY}}\bar{u}_y{}^2$$

$$+ \frac{\Lambda_{wy1}^2}{B_0 R_{BY}}\bar{w}_y{}^2 + \frac{(\Lambda_{uz1}^2 - \Omega_s^2)}{A_0 R_{AZ}}\bar{u}_z{}^2 + \frac{(\Lambda_{vz1}^2 - \Omega_s^2)}{B_0 R_{BZ}}\bar{v}_z{}^2\} \qquad (119)$$

where

$$\kappa_5|_E \leq \kappa_3|_E \qquad (120)$$

Hence, if $\kappa_5|_E$ is positive definite the equilibrium point is asymptotically stable. We note that $\kappa_5|_E$ can be written as the sum of three quadratic forms, each of which must be positive definite. Denoting these forms by $\kappa_{51}|_E$, $\kappa_{52}|_E$, $\kappa_{53}|_E$ and their associated Hessian matrices by $[\mathcal{H}_{51}]_E$, $[\mathcal{H}_{52}]_E$, $[\mathcal{H}_{53}]_E$, respectively, we obtain

$$[\mathcal{H}_{51}]_E = \frac{1}{2}\begin{bmatrix} \dfrac{(\Lambda^2_{vx1}-\Omega^2_s)}{A_0R_{AX}} & 0 \\ 0 & \dfrac{(\Lambda^2_{uy1}-\Omega^2_s)}{B_0R_{BY}} \end{bmatrix} \tag{121a}$$

$$[\mathcal{H}_{52}]_E = \frac{\Omega^2_s}{2A}\begin{bmatrix} C(C-A) & -C & -C \\ -C & \dfrac{\Lambda^2_{wx1}A}{\Omega^2_s A_0R_{AX}} + 1 & 1 \\ -C & 1 & (\dfrac{\Lambda^2_{uz1}}{\Omega^2_s} - 1)\dfrac{A}{A_0R_{AZ}} + 1 \end{bmatrix} \tag{121b}$$

$$[\mathcal{H}_{53}]_E = \frac{\Omega^2_s}{2B}\begin{bmatrix} C(C-B) & C & C \\ C & \dfrac{\Lambda^2_{wy1}B}{\Omega^2_s B_0R_{BY}} +1 & 1 \\ C & 1 & (\dfrac{\Lambda^2_{vz1}}{\Omega^2_s} - 1)\dfrac{B}{B_0R_{BZ}} + 1 \end{bmatrix} \tag{121e}$$

An application of Sylvester's criterion to matrices (121), yields the following stability criteria

$$\Lambda^2_{vx1} > \Omega^2_s \quad , \quad \Lambda^2_{uy1} > \Omega^2_s \tag{122a}$$

$$C > A$$

$$\left(\frac{\Lambda_{wx1}}{\Omega_s}\right)^2 > \frac{A_0R_{AX}}{C - A} \tag{122b}$$

$$\left(\frac{\Lambda_{uz1}}{\Omega_s}\right)^2 > 1 + \frac{(\Lambda_{wx1}/\Omega_s)^2 \; A_0R_{AZ}}{(C-A)(\Lambda_{wx1}/\Omega_s)^2 - A_0R_{AX}}$$

and

$$C > B$$

$$\left(\frac{\Lambda_{wy1}}{\Omega_s}\right)^2 > \frac{B_0 R_{BY}}{C - B} \qquad (122c)$$

$$\left(\frac{\Lambda_{vz1}}{\Omega_s}\right)^2 \quad 1 + \frac{(\Lambda_{wy1}/\Omega_s)^2 \, B_0 R_{BZ}}{(C-B)(\Lambda_{wy1}/\Omega_s)^2 - B_0 R_{BY}}$$

respectively. From our previous discussion we conclude that inequalities (122a) are always satisfied. Furthermore, we note that inequalities (122b) and (122c) possess identical forms. In view of that, we shall establish stability criteria using inequalities (122b) and replace the parameters A, A_0, R_{AX}, R_{AZ}, Λ_{wx1}/Ω_s and Λ_{uz1}/Ω_s by B, B_0, R_{BY}, R_{BZ}, Λ_{wy1}/Ω_s and Λ_{vz1}/Ω_s respectively, to derive criteria valid for (122c). For convenience, inequalities (122b) are written in the slightly different form

$$C > A \qquad (123a)$$

$$\frac{\Omega_s}{\Lambda_{wx1}} < \left[\frac{C_0/A_0 + R_{AX} - 1 - R_{AZ}}{R_{AX}}\right]^{1/2} \qquad (123b)$$

$$\frac{\Omega_s}{\Lambda_{uz1}} < \left[1 + \frac{R_{AZ}}{C_0/A_0 - R_{AZ} - 1 + R_{AX}(1 - \Omega_s^2/\Lambda_{wx1}^2)}\right]^{-1/2} \qquad (123c)$$

Three major conclusions can be drawn from inequalities (123):

(a) For spin stabilization the spinning motion should be imparted about the axis of maximum moment of inertia.

(b) Spin stabilization is possible if the spin ratios Ω_s/Λ_{wx1}

and Ω_s/Λ_{uz1} satisfy inequalities (123b) and (123c), which involve the system parameters R_{AX}, R_{AZ} and C_0/A_0. In addition, the frequency ratio Ω_s/Λ_{uz1} should not exceed unity.

(c) A satellite which is stable without radial rods remains stable if radial rods are added.

To verify the last statement, we recall that Λ_{wx1} represents the first natural frequency of the out-of-plane vibration of a rotating rod and it must be greater than Ω_s, so that $\Omega_s/\Lambda_{wx1} < 1$. In addition, for a satellite with no radial rods, we find from inequality (123a), that for stability we should have $C_0/A_0 > 1 + R_{AZ}$. Using these results, we see that inequality (123b) yields a less stringent criterion as the right side of (123b) is always greater than unity. Moreover, for any value of R_{AX} other than zero, inequality (123c) is less restrictive than the same inequality with $R_{AX} = 0$.

We note that, by contrast with inequalities (95), the evaluation of criteria (123) requires much less numerical work. In particular, for inequalities (95) we must obtain a complete solution of the eigenvalue problem (106) consisting of the n frequencies ${}^n\Lambda_{wxi}$ and eigenvectors $\{a^{(i)}\}$, whereas inequalities (123) require only the first natural frequency Λ_{wx1} of the rotating rod.

Numerical Results

The general solution of the stability problem of a rigid satellite with three (or less) pairs of uniform rods has been programmed for digital computation, and a numerical solution obtained on an IBM 360 computer. Results are presented for the criteria developed using both modal analysis and integral coordinates. For the numerical study it is assumed that rods x and z have equal mass and stiffness properties, and, in addition, the rigid body dimensions h_x and h_z are equal (see later statement concerning rods y). The above restrictions are placed only on the numerical solution to facilitate the presentation of data; there are no such restrictions placed on either the problem formulation or computer program. In the figures presented, the results obtained using modal analysis are represented by the dashed lines and those obtained using integral coordinates by solid lines. Figure 3 shows the value of the ratio $\Omega_s/(\Lambda_{wx1})_{NR}$ vs Ω_s/Λ_{wx1}, where $(\Lambda_{wx1})_{NR}$ is the first natural frequency of the nonrotating rod, obtained by setting $\Omega_s = 0$. The first natural frequency of the rotating rod is denoted by Λ_{wx1}. The quantity $HX = h_x/\ell_x$ plays the role of a parameter. This figure enables us to make use of the parameter plots of Fig. 4 without having to solve the eigenvalue problem for the rotating rods, where Fig. 4 shows the spin ratio Ω_s/Λ_{wx1} required for stability as a function of $(C_0/A_0) - R_{AZ}$, with R_{AX} as a parameter. The region below the appropriate curve is stable. The curve shows that for $(C_0/A_0) - R_{AZ} = 1$ the allowable spin ratio is equal to unity, and for $(C_0/A_0) - R_{AZ} > 1$ no instability exists. We note in Fig. 3 that the ratio Ω_s/Λ_{wx1} is always less than unity. The extent to which

Ω_s/Λ_{wx1} is less than unity depends on the parameter HX, in the sense that if HX increases the ratio Ω_s/Λ_{wx1} decreases. Hence, in Fig. 4 all values of Ω_s/Λ_{wx1} greater than unity are said to be dynamically impossible. However, the dynamically impossible region may include values of Ω_s/Λ_{wx1} considerably less than unity as shown in Fig. 3. It is reiterated again that Figs. 3 and 4 are to be used together. Namely, starting with a value of $\Omega_s/(\Lambda_{wx1})_{NR}$, Fig. 3 gives Ω_s/Λ_{wx1}, which is then used in Fig. 4. It should be noted that Figs. 3 and 4 present a complete stability analysis for a satellite which radial rods alone. Figures 5 through 8 show the allowable spin ratio Ω_s/Λ_{uz1} for stability as a function of R_{AZ}, with the length ratio ℓ_x/ℓ_z as a parameter. The region below the appropriate curve is stable. These curves show that the allowable spin ratio Ω_s/Λ_{uz1} must be lower than unity; the extent to which it must be lower than unity depends on the system parameters. It should be noted from Figs. 6 through 8 that the most restrictive region of stability is associated with the parameter $\ell_x/\ell_z = 0$, namely the case in which there are no radial rods. As noted earlier, any stable satellite possessing axial rods alone will remain stable with the addition of radial rods. Indeed the addition of radial rods increases the region of stability significantly and for length ratios $\ell_x/\ell_z > 10$ the allowable spin ratio is very near unity. Figure 9 shows the effect of changing the rigid body inertia ratio C_0 / A_0 on the allowable spin ratio for a fixed value of the length ratio ℓ_x/ℓ_z. Again the region below the appropriate curve is stable. As expected, an increase in C_0/A_0 increases the stable region. Figure 10 shows the effect of changes in the parameter HZ, where HZ = h_z/ℓ_z. Again the region below the appropriate curve is stable. Figure 10 also shows that increasing HZ yields a slight increase in the stability region. Figures 5 through 10 represent criteria determined by inequalities (95) and (123c) and are due to the addition of z rods.

For comparison purposes, a problem which can be considered as a special case of the present one, in the sense that it considers only spin axis rods, has been considered; this is the problem investigated in Ref. 16. Inequality (123c) for the case where R_{AX} equals zero yields the appropriate stability criteria. Results using this criteria as well as results from Ref. 16 are presented in Fig. 11. The results of Ref. 16 working with density functions are more restrictive than those of the present investigation.

It should be noted that diagrams identical in every respect to Figs. 3 through 11 but with Λ_{vz1}, Λ_{wy1}, B_0, R_{BY}, R_{BZ} and ℓ_y replacing Λ_{uz1}, Λ_{wx1}, A_0, R_{AX}, R_{AZ} and ℓ_x, respectively, can be obtained from inequalities (100) and (128c).

Summary and Conclusions

The mathematical formulation associated with the problem of the stability of motion of a satellite consisting of a main rigid body and three (or less) pairs of flexible rods has been completed. The rods are capable of flexure in two orthogonal directions. Whereas the rotational motion of the body is described by generalized coordinates depending on time alone, the elastic displacements of the rods depend both on spatial position and time. Because of the elastic motion of the rods, the center of mass of the body is shifting continuously relative to the main rigid body. These displacements, however, do not add degrees of freedom since they can be expressed in terms of integrals involving the elastic displacements. Assuming no external torques, there exist motion integrals in the form of momentum integrals. These integrals can be regarded as constraint equations relating the system velocities.

The Liapunov direct method has been chosen for the stability analysis because it is likely to yield results which can be interpreted more readily than those obtained by a purely numerical integration of the equations of motion. Since the elastic vibrations result in energy dissipation, it is shown that the equilibrium position is asymptotically stable if the Hamiltonian is positive definite and unstable if it can take negative values in the neighborhood of the equilibrium. Determining the sign definiteness of the Hamiltonian is complicated by the fact that the Hamiltonian contains spatial derivatives of the elastic displacements. Two methods have been presented to deal with this problem. The first, the standard modal analysis in conjunction with series truncation, develops criteria which are expressed in terms infinite series associated with the natural modes and frequencies of the elastic rods. The second, the method integral coordinates yields closed-form stability criteria involving the system parameters such as the body moments of inertia, the length and mass distribution of the rods, the lowest natural frequencies of the rods, and the satellite spin velocity. The advantage of the method of integral coordinates is illustrated by the relative ease with which closed-form stability criteria are developed and by the amount of information which can be extracted from their ready physical interpretation. In particular, the analysis shows that, for stability, the spinning motion is to be imparted about the axis of maximum moment of inertia and that the allowable spin ratios Ω_s/Λ_{wx1}, Ω_s/Λ_{wy1}, Ω_s/Λ_{uz1} and Ω_s/Λ_{vz1} are determined by the system parameters. The first is recognized as the "greatest moment of inertia" criterion. Moreover, the spin ratios Ω_s/Λ_{uz1} and Ω_s/Λ_{vz1} should not be merely lower than unity (as they should be in the case of simple harmonic excitation of the rods to prevent reasonance), but they are further restricted by the system parameters. It is also shown that a stable spinning satellite which does not contain radial rods will remain stable if radial rods are added.

Appendix A

The out-of-plane vibration of a rotating fixed-free rod, attached to a hub of radius h_x, is subject to an axial centrifugal force and its eigenvalue problem is defined by the differential equation (68) and the associated boundary conditions (69). The first natural frequency for such a rod is always greater than the rate of rotation Ω_s. This can be shown to be true for a rod of arbitrary mass and stiffness distribution. To prove this statement we recall Rayleigh's quotient

$$R(\phi) = \frac{\int_D \phi \mathcal{L}[\phi]\, dD}{\int_D \phi M[\phi]\, dD} \tag{A1}$$

where, for the problem at hand, the operators $\mathcal{L}$ and M are given by

$$\mathcal{L} = \frac{d^2}{dx^2}\left[EI_{vx}(x)\frac{d^2}{dx^2}\right] - \frac{d}{dx}\left[P_x(x)\frac{d}{dx}\right]$$

$$M = \rho_x(x) \tag{A2}$$

The domain of extension of the rod is $D : h_x \leq x \leq h_x + \ell_x$ and the centrifugal force $P_x(x)$ has the expression

$$P_x(x) = \Omega_s^2 \int_x^{h_x+\ell_x} \rho_x(\xi)\,\xi\, d\xi \tag{A3}$$

From the properties of Rayleigh's quotient, we recall that

$$R(\phi) \geq \Lambda_{vx1}^2 \tag{A4}$$

where Λ_{vx1} represents the first natural frequency of vibration associated with v_x, and ϕ represents any comparison function. Furthermore, the equality sign in (A4) holds only if ϕ represents the eigenfunction associated with the first natural frequency. Letting ϕ_1 represent the eigenfunction associated with Λ^2_{vx1}, integrating by parts, and considering conditions (69), Eq.(A1) yields

$$\Lambda^2_{vx1} = \frac{\int_{h_x}^{h_x+\ell_x} EI_{vx}(x) \left(\frac{d^2\phi_1}{dx^2}\right)^2 dx + \int_{h_x}^{h_x+\ell_x} P_x(x) \left(\frac{d\phi_1}{dx}\right)^2 dx}{\int_{h_x}^{h_x+\ell_x} \rho_x(x) \phi_1^2 \, dx} \tag{A5}$$

We note that Eq. (A5) can be written as

$$\Lambda^2_{vx1} = \frac{V_{EI}(\phi_1)}{T(\phi_1)} + \frac{V_P(\phi_1)}{T(\phi_1)} \tag{A6}$$

where $V_{EI}(\phi_1)$ represents the potential energy due to bending, $V_P(\phi_1)$ the potential energy due to the axial centrifugal force and $T(\phi_1)$ a reference kinetic energy (see Ref. 17, Sec. 6.4). Hence, Rayleigh's quotient can be expressed as the sum of two independent terms, one representing the bending energy, and the other corresponding to the energy associated with the centrifugal force.

Due to the above result, we consider two problems related to the problem above. The first, a nonrotating fixed-free rod with mass and stiffness distribution identical to that of the rotating rod and the second, a rotating fixed-free string with mass density identical to that of the rotating rod but with zero flexural stiffness. Both problems are defined over the domain D. Writing Rayleigh's quotient for each of these problems we obtain

$$R_r(\phi) = \frac{V_{EI}(\phi)}{T(\phi)} \geq \Lambda_{r1}^2$$
$$R_s(\phi) = \frac{V_P(\phi)}{T(\phi)} \geq \Lambda_{s1}^2 \tag{A7}$$

where the subscripts r and s refer to the nonrotating rod and the rotating string, respectively. Using as a comparison function the eigenfunction ϕ_1, inequalities (A7) take the form

$$\frac{V_{EI}(\phi_1)}{T(\phi_1)} > \Lambda_{r1}^2$$
$$\frac{V_P(\phi_1)}{T(\phi_1)} > \Lambda_{s1}^2 \tag{A8}$$

and recalling Eq.(A6), we obtain*

$$\Lambda_{vx1}^2 > \Lambda_{r1}^2 + \Lambda_{s1}^2 \tag{A9}$$

Inequality (A9) indicates that the square of the first natural frequency of the rotating rod is always greater than the sum of the squares of the first natural frequencies of the nonrotating rod and the rotating string, respectively.

Let us give further consideration to the first natural frequency of the string, Λ_{s1}.

---

* This result is due to a Theorem by Southwell. See Ref. 18.

The differential equation for the string is given by

$$-\frac{d}{dx}\left[P_x(x)\frac{dv_x}{dx}\right] = \rho_x(x)\,\Lambda_{s1}^2\,v_x \tag{A10}$$

where Eq. (A10) is subject to the boundary conditions

$$v_x(h_x) = 0 \quad , \quad P_x(x)\left.\frac{dv_x(x)}{dx}\right|_{x=h_x+\ell_x} = 0 \tag{A11}$$

For comparison purposes, let us define the eigenvalue problem for a string of length $h_x + \ell_x$ rotating about the point $x = 0$ with angular rate Ω_s. The mass distribution for the string is given by

$$\hat{\rho}_x(x) = \begin{cases} \rho_x(h_x) & o \leq x \leq h_x \\ \rho_x(x) & h_x < x \leq h_x+\ell_x \end{cases} \tag{A12}$$

Denoting the transverse displacement of the string by $\hat{v}_x$, the associated differential equation is

$$-\frac{d}{dx}\left[P_x(x)\frac{d\hat{v}_x(x)}{dx}\right] = \hat{\rho}_x(x)\,\hat{\Lambda}_{s1}^2\,\hat{v}_x(x) \tag{A13}$$

where Eq. (A13) is subject to the boundary conditions

$$\hat{v}_x(0) = 0 \quad ; \quad P_x(x)\left.\frac{d\hat{v}_x(x)}{dx}\right|_{x=h_x+\ell_x} = 0 \tag{A14}$$

Recalling that $P_x(x) = \Omega_s^2 \int_x^{h_x+\ell_x} \hat{\rho}_x(\xi)\xi\, d\xi$, it is not difficult to show that Eq. (A13) subject to conditions (A14) admits a solution of the form

$$\hat{v}_x(x) = x/(h_x+\ell_x) \tag{A15}$$

corresponding to the frequency

$$\hat{\Lambda}^2_{s1} = \Omega^2_s \tag{A16}$$

From Eqs. (A15) and (A16), we conclude that a string rotating about an axis through its fixed end has a rigid body mode and a corresponding first natural frequency equal to the spin rate Ω_s. We wish to show that the first natural frequency of a rotating string fixed to a hub of radius h_x must always exceed Ω_s. To this end, we consider Rayleigh's quotient for Eq. (A13). Using Eq. (A16), we obtain

$$\hat{R}_s(\phi) = \frac{\hat{V}_P(\phi)}{\hat{T}(\phi)} \geq \Omega^2_s \tag{A17}$$

Consider as an admissible function* ϕ in Eq. (A17) the following function

$$\phi = \begin{cases} 0 & o \leq x \leq h_x \\ \phi_{s1} & h_x \leq x \leq h_x + \ell_x \end{cases} \tag{A18}$$

where ϕ_{s1} represents the first eigenfunction for Eq. (A10) subject to conditions (A11). From Eq. (A17) we obtain

---

* See Reference 19, Chapter VI, Sec. 7.1.

$$\hat{R}_s(\phi) = \frac{\int_0^{h_x+\ell_x} P_x(x)\left(\frac{d\phi}{dx}\right)^2 dx}{\int_0^{h_x+\ell_x} \hat{\rho}_x(x) \quad {}^2\, dx} > \Omega_s^2 \tag{A19}$$

However, using the definition of ϕ given in Eq. (A18), we find

$$\hat{R}_s(\phi) = \frac{\int_{h_x}^{h_x+\ell_x} P_x(x)\left(\frac{d\phi_{s1}}{dx}\right)^2 dx}{\int_{h_x}^{h_x+\ell_x} \rho_x(x)\ \phi_{s1}^2\, dx} = \Lambda_{s1}^2 \tag{A20}$$

Combining the results of Eqs. (A19) and (A20), we have

$$\Lambda_{s1}^2 > \Omega_s^2 \tag{A21}$$

Therefore, we can state that the first natural frequency for a rotating string attached to a hub of radius h_x is always greater than the corresponding frequency for a string attached to the axis of rotation. Using this result in inequality (A9), we obtain

$$\Lambda_{vx1}^2 > \Lambda_r^2 + \Omega_s^2 \tag{A22}$$

which completes the proof.

Appendix B

The computer program consists of a main program and four subroutines. Listings for the main program and one of the subroutines are provided. The three remaining subroutines are IBM SSP subroutines ARRAY, EIGEN and NROOT. These subroutines are readily available.

The main program is capable of performing a stability analysis based on criteria established for either the normal mode analysis or the integral coordinate analysis. The program solves the eigenvalue problem of Eq.(106) and uses this, along with various input parameters, to establish stability bounds for either the normal modes or integral coordinates method, depending on the input parameters. In the following the input and output parameters are listed with accompanying explanatory statements.

```
C
C  ....................................................................
C
C     MAIN PROGRAM
C
C     PURPOSE
C        TO TEST THE STABILITY OF A SPINNING SATELLITE WITH THREE OR
C        LESS PAIRS OF FLEXIBLE RODS USING EITHER THE METHOD OF NORMAL
C        MODES OR THE METHOD OF INTEGRAL COORDINATES
C
C     DESCRIPTION OF INPUT PARAMETERS
C        RHOR - RATIO OF THE MASS DENSITY OF ROD X TO THAT OF ROD Z
C        EIR - RATIO OF THE FLEXURAL STIFFNESS OF ROD X TO THAT OF ROD Z
C        HR - RATIO OF THE ROD ATTACHMENT DISTANCE IN THE X DIRECTION
C             TO THAT IN THE Z DIRECTION
C        HZ - RATIO OF THE ROD ATTACHMENT DISTANCE IN THE Z DIRECTION
C             TO THE LENGTH OF THE Z ROD
C        SPIN - RATIO OF THE SPIN RATE OMEGA-S TO THE FIRST NATURAL
C               FREQUENCY OF THE MOTION OF THE Z ROD IN THE X DIRECTION
C        NOX - NUMBER OF ODD MODES ASSUMED FOR THE X RODS
C        NOZ - NUMBER OF ODD MODES ASSUMED FOR THE Z RODS
C        N - NUMBER OF ASSUMED MODES IN THE EIGENVALUE PROBLEM FOR THE
C            OUT-OF-PLANE VIBRATION OF THE X ROD
C        IX - CONFIGURATION INDICATOR. IF IX = 0, NO X RODS ARE PRESENT,
C             IF IX = 1, X RODS ARE PRESENT
C        IZ - CONFIGURATION INDICATOR. IF IZ = 0, NO Z RODS ARE PRESENT,
C             IF IZ = 1, Z RODS ARE PRESENT
C        JJ - RUN TYPE INDICATOR. IF JJ = 0, PROGRAM RUNS PARAMETER
C             STUDY OF SPIN VS. RAZ FOR FIXED VALUES OF LR. JJ=1,
C             PROGRAM RUNS PARAMETER STUDY OF SPIN VS. COAC FOR FIXED
C             VALUES OF RAX
C        II - INDICATOR FOR METHOD OF ANALYSIS. IF II = 0, THE PROGRAM
C             TESTS FOR STABILITY USING INTEGRAL COORDINATES, IF II = 1,
C             THE PROGRAM TESTS FOR STABILITY USING NORMAL MODES
C        RAZ - RATIO OF THE MOMENT OF INERTIA OF THE Z RODS ABOUT THE
C              X AXIS TO THE RIGID BODY MOMENT OF INERTIA AC. NOTE, WHEN
C              JJ = 1, VALUES OF RAX ARE ENTERED FOR THIS VARIABLE
C        COAO - RATIO OF THE RIGID BODY INERTIAS CO AND AC
C        LR - RATIO OF THE LENGTHS OF RODS X AND Z.
C
C     OUTPUT PARAMETERS
C        T1 - TESTING FUNCTION ASSOCIATED WITH THE GREATEST MOMENT OF
C             INERTIA RULE - MUST BE POSITIVE FOR STABILITY
C        T2 - TESTING FUNCTION ASSOCIATED WITH X RODS USING NORMAL MODE
C             ANALYSIS - MUST BE POSITIVE FOR STABILITY
C        T3 - TESTING FUNCTION ASSOCIATED WITH X AND Z RODS USING NORMAL
C             MODE ANALYSIS - MUST BE POSITIVE FOR STABILITY
C        T4 - TESTING FUNCTION ASSOCIATED WITH X RODS USING INTEGRAL
C             COORDINATES - MUST BE POSITIVE FOR STABILITY
C        T5 - TESTING FUNCTION ASSOCIATED WITH X AND Z RODS USING
C             INTEGRAL COORDINATES - MUST BE POSITIVE FOR STABILITY
C        SPIN1 - RATIO OF THE SPIN RATE OMEGA-S TO THE FIRST NATURAL
C                FREQUENCY OF THE X ROD IN THE Z DIRECTION
C
```

```
C
C    ...................................................................
C
      DIMENSION DEL(10,10),AK(10,10),AM(10,10),ALUZ(10),ALWX(10),
     $ PAL(10),BLWX(10),X(10),BETAL(10),SX(10),SZ(10),RX(10),VAL(10),
     $VEC(10,10),Y(200),AI(10,10)
   10 FORMAT(5F10.5)
   20 FORMAT(8I5)
   30 FORMAT(2E20.7)
   40 FORMAT(3F10.5)
      PHI(A,C)= (SIN(A)-SINH(A))*(COS(A*C)-COSH(A*C))-(COS(A)+COSH(A))*
     $(SIN(A*C)+SINH(A*C))
      FUNCT(A,B,C)=(1.-C**2+2.*HX*(1.-C))*PHI(A,C)*PHI(B,C)
      REAL LR
   50 READ(5,40,END=370) RAZ,COAO,LR
      READ(5,10) RHOR,EIR,HR,HZ,SPIN
      READ(5,20) NCX,NCZ,MA,N,IX,IZ,JJ,II
      ICOUNT=0
      IF(LR.EQ.0.0) GO TO 60
      HX=HZ*HR/LR
      IF(IZ.EQ.0) RAX=RAZ
      GO TO 70
   60 HX=0.0
   70 CONTINUE
      TAUX=HX**2+HX+1./3.
      TAUZ=HZ**2+HZ+1./3.
      IF(IZ.EQ.0) GO TO 71
      RAX=RAZ*(LR**3)*RHOR*TAUX/TAUZ
      GO TO 72
   71 RAZ=0.0
   72 CONTINUE
      ALR=LR
      IF(LR.EQ.0.0)ALR=1.0
      BLWX(1)=SQRT(1./RHOR)*SQRT(EIR)*(1./ALR)**2
      CALL RTS(N,BETAL,50,1.E-6)
      DO 80 I=1,N
      X(I)=BETAL(I)
      RX(I)=SQRT(2.)*(HX*X(I)*(SIN(X(I))-SINH(X(I)))-(COS(X(I))+
     $COSH(X(I))))/X(I)/X(I)/SIN(X(I))/SINH(X(I))
      ALUZ(I)=X(I)**2/X(1)**2
      ALWX(I)=BLWX(1)*X(I)**2/X(1)**2
   80 SZ(I)=SQRT(2.)*(HZ*X(I)*(SIN(X(I))-SINH(X(I)))-(COS(X(I))+
     $COSH(X(I))))/X(I)/X(I)/SIN(X(I))/SINH(X(I))
      DO 90 I=1,N
      DO 90 J=1,N
   90 DEL(I,J)=0.0
      DO 100 I=1,N
  100 DEL(I,I)=1.0
  110 CONTINUE
      Y(1)=0.0
      XMA=MA
      DO 120 I=2,MA
  120 Y(I)=Y(I-1)+1.0/XMA
```

```
      SPXN=SPIN/BLWX(1)
      DO 150 I=1,N
      DO 150 J=I,N
      EVEN=0.0
      ODD=0.0
      MB=MA-2
      DO 130 K=2,MB,2
130   EVEN=EVEN+FUNCT(X(I),X(J),Y(K))
      MC=MA-1
      DO 140 K=3,MC,2
140   ODD=ODD+FUNCT(X(I),X(J),Y(K))
      AI(I,J)=(Y(2)-Y(1))/3.*(FUNCT(X(I),X(J),Y(1))+2.*ODD+4.*EVEN+
     $FUNCT(X(I),X(J),Y(MA)))
      AM(I,J)=DEL(I,J)
150   AK(I,J)=((X(I)/X(1))**4)*DEL(I,J)+((1./2.)*SPXN**2)*X(I)*X(J)*
     $AI(I,J)/SIN(X(I))/SIN(X(J))/SINH(X(I))/SINH(X(J))
      DO 160 J=1,N
      DO 160 I=J,N
      AK(I,J)=AK(J,I)
160   AM(I,J)=AM(J,I)
      CALL ARRAY(2,N,N,10,10,AK,AK)
      CALL ARRAY(2,N,N,10,10,AM,AM)
      CALL NROOT(N,AM,AK,VAL,VEC)
      WRITE(6,170)
170   FORMAT(//'   THE OUT OF PLANE EIGENVALUES ARE')
      DO 180 I=1,N
180   PAL(I)=SQRT(1./VAL(I))/SPXN
      DO 190 I=1,N
190   ALWX(I)=PAL(I)*SPIN
      WRITE(6,30)(PAL(I),I=1,N)
200   WRITE(6,210)
210   FORMAT(//'    THE UX OUT OF PLANE VECTORS ARE')
      CALL ARRAY(1,N,N,10,10,VEC,VEC)
      WRITE(6,30)((VEC(I,J),I=1,N),J=1,N)
      DO 220 I=1,NOX
      SX(I)=0.0
      DO 220 K=1,N
220   SX(I)=SX(I)+VEC(K,I)*RX(K)
      WRITE(6,280)
      WRITE(6,300) RAX,RAZ,COAO,HZ,RHOR,EIR,HR,LR,N,NOX,NOZ
      T1=COAO-1.+RAX-RAZ
      IF(T1.LT.0.0) GO TO 221
      GO TO 223
221   WRITE(6,222)
222   FORMAT(//' THE GREATEST MOMENT OF INERTIA RULE HAS BEEN VIOLATED')
      GO TO 50
223   CONTINUE
      D=2.*(RAX/TAUX)*SX(1)**2
      E=0.
      DO 230 I=2,NOX
230   E=E+2.*(RAX/TAUX)*(SX(I)*SPIN/ALWX(I))**2
      F=2.*(RAZ/TAUZ)*SZ(1)**2
      G=0.
```

```
      DO 240 I=2,NOZ
240   G=G+2.*(RAZ/TAUZ)*SZ(I)**2/((ALUZ(I)/SPIN)**2-1.)
      IF(IX.EQ.0) GO TO 250
      T2=SQRT((T1-E)/D)-SPIN/ALWX(1)
      T4=SQRT(T1/RAX)-SPIN/ALWX(1)
      IF(IZ.EQ.0) GO TO 270
      GO TO 260
250   T2=0.0
      T4=0.0
260   CONTINUE
      T3=1./SQRT(1.+F/(T1-G-E-D*(SPIN/ALWX(1))**2))-SPIN
      T5=1./SQRT(1.+RAZ/(T1-RAX*(SPIN/ALWX(1))**2))-SPIN
      GO TO 271
270   T3=0.0
      T5=0.0
271   CONTINUE
      SPIN1=1./RAL(1)
280   FORMAT(//'      RAX          RAZ         CC/AC          HZ          RHOR         E
     $IR         HR          LR          N     NOX   NOZ')
      WRITE(6,290)
290   FORMAT(//'      T1           T2           T3           T4           T5          SP
     $IN         SPIN1 ')
      WRITE(6,301) T1,T2,T3,T4,T5,SPIN,SPIN1
300   FORMAT(9F10.5,3I5)
301   FORMAT(7F10.5)
      IF(II.EQ.0) GO TO 302
      GO TO 303
302   T6=T4
      T7=T5
      GO TO 304
303   T6=T2
      T7=T3
304   CONTINUE
      IF(T6.LT.0.0.AND.ICOUNT.EQ.2) GO TO 360
      IF(T7.LT.0.0.AND.ICOUNT.EQ.2) GO TO 360
      IF(T6.LT.0.0) GO TO 310
      IF(T7.LT.0.0) GO TO 310
      IF(T6.GT.0.0.AND.ICOUNT.NE.0) GO TO 330
      IF(T7.GT.0.0.AND.ICOUNT.NE.0) GO TO 330
      SPIN=SPIN+.5
      IF(SPIN.GT.10.) GO TO 50
      GO TO 110
310   ICOUNT=1
      SPIN=SPIN-.1
      GO TO 110
320   IF(IX.EQ.0) GO TO 350
      RAZ=RAZ+.2
      ICOUNT=0
      IF(RAZ.GT.1.1) GO TO 50
      GO TO 70
330   ICOUNT=2
      SPIN=SPIN+.01
      GO TO 110
```

```
340   COAO=COAO+.1
      ICOUNT=0
      IF(COAO.GT.1.0) GO TO 50
      GO TO 70
350   RAZ=RAZ+(COAO-1.)/5.
      ICOUNT=0
      GO TO 70
360   IF(JJ.EQ.1) GO TO 340
      GO TO 320
370   CALL EXIT
      END
```

```
C
C    ..................................................................
C
C       SUBROUTINE RTS
C
C       USAGE
C          CALL RTS(N,RES,ITER,TOL)
C
C       PURPOSE
C          TO FIND THE ROOTS OF THE EQUATION COS(RES)COSH(RES)=-1
C
C       DESCRIPTION OF PARAMETERS
C          N - NUMBER OF ROOTS DESIRED
C          RES - RESULTING ROOTS
C          ITER -  NUMBER OF ITERATIONS ALLOWED
C          TOL- TOLERANCE ALLOWABLR ON THE VALUE OF THE ROOTS
C
C    ..................................................................
C
      SUBROUTINE RTS(N,RES,ITER,TOL)
      DIMENSION RES(25)
      J=1
      I=1
      PI=3.1415927
      X1=PI/2.
    2 CX=COS(X1)
      CSX=COSH(X1)
      SX=SIN(X1)
      SSX=SINH(X1)
      F=CX*CSX+1.
      FD=CX*SSX-SX*CSX
      X2=X1-F/FD
      WRITE(6,101)X2
  101 FORMAT(E20.5)
      DIFF=X2-X1
      IF(ABS(DIFF)-TOL)20,20,10
   10 X1=X2
      J=J+1
      IF(J.GT.ITER) GO TO 15
      GO TO 2
   15 WRITE(6,100)
      RETURN
   20 RES(I)=X2
      WRITE(6,102)RES(I)
  102 FORMAT(10X,E20.5)
      I=I+1
      IF(I.GT.N)RETURN
      X1=X1+PI
      J=1
      GO TO 2
  100 FORMAT(23H NO CONVERGENCE IN RTS    )
      END
```

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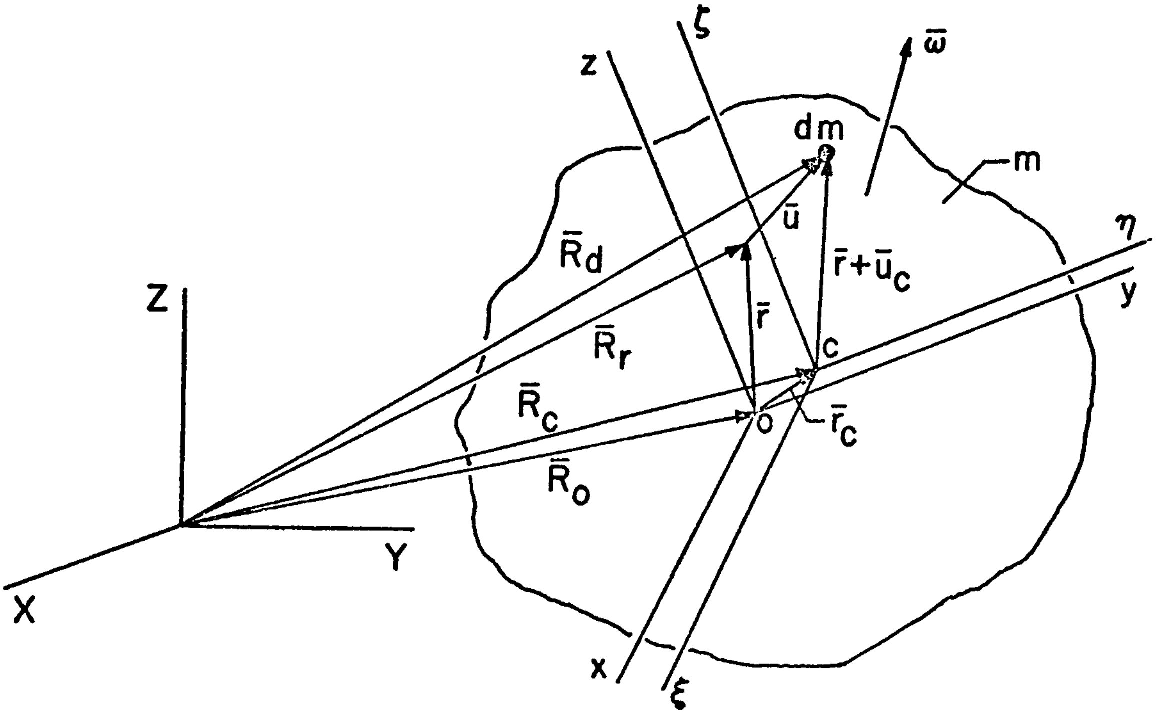
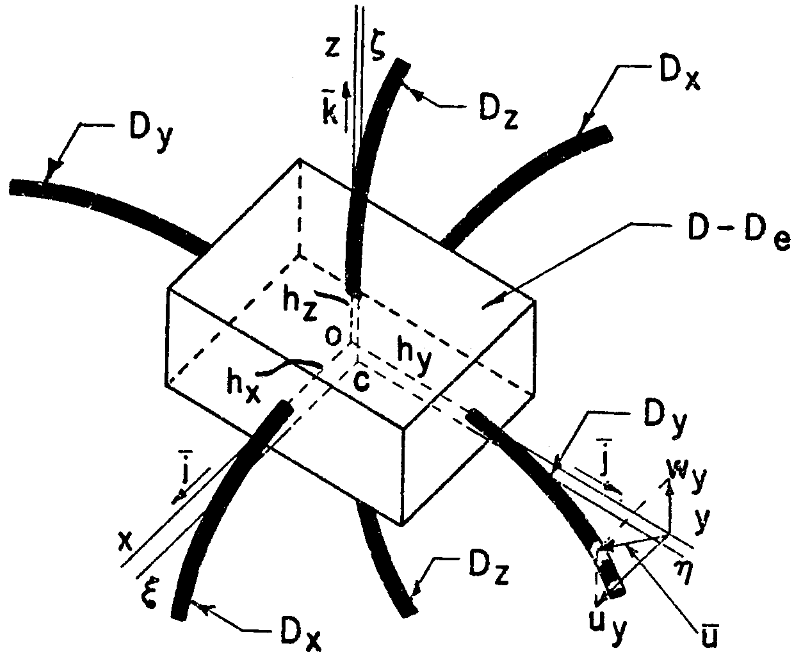
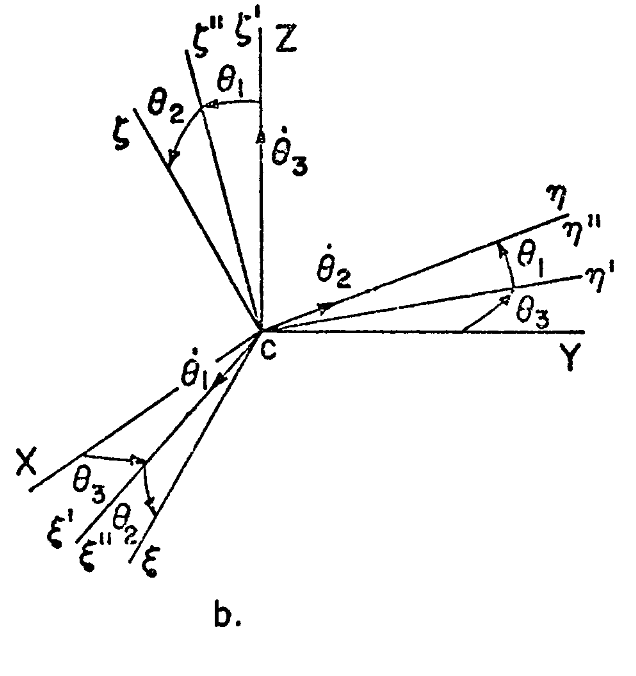


Figure 1 — The Flexible Body in an Inertial Space



a.



b.

Figure 2a — The Flexible Satellite
2b — The Satellite Rotational Motion

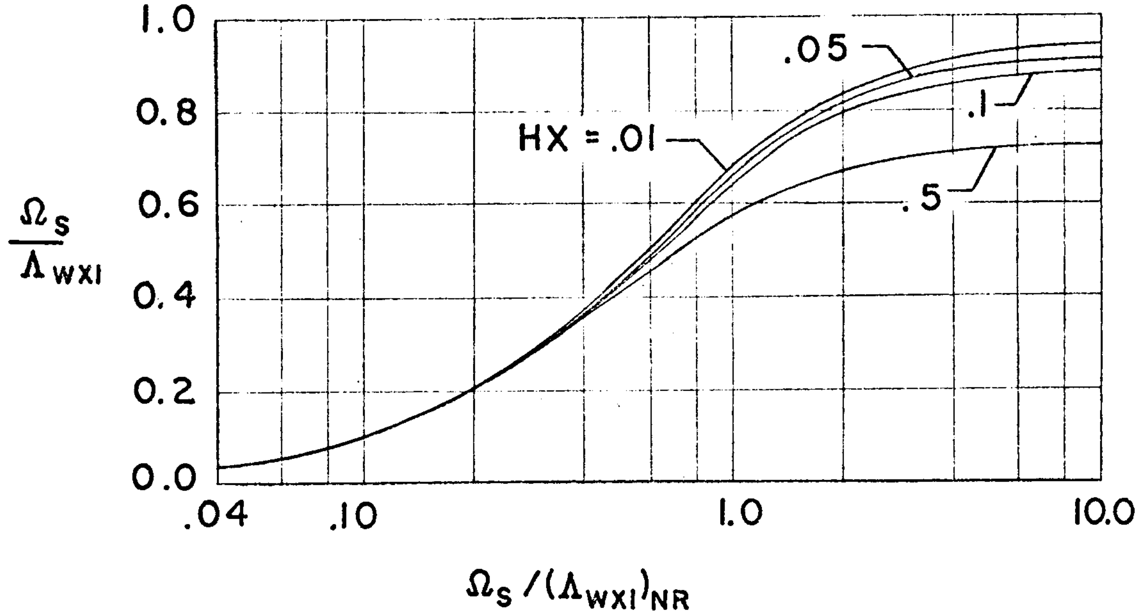


Figure 3

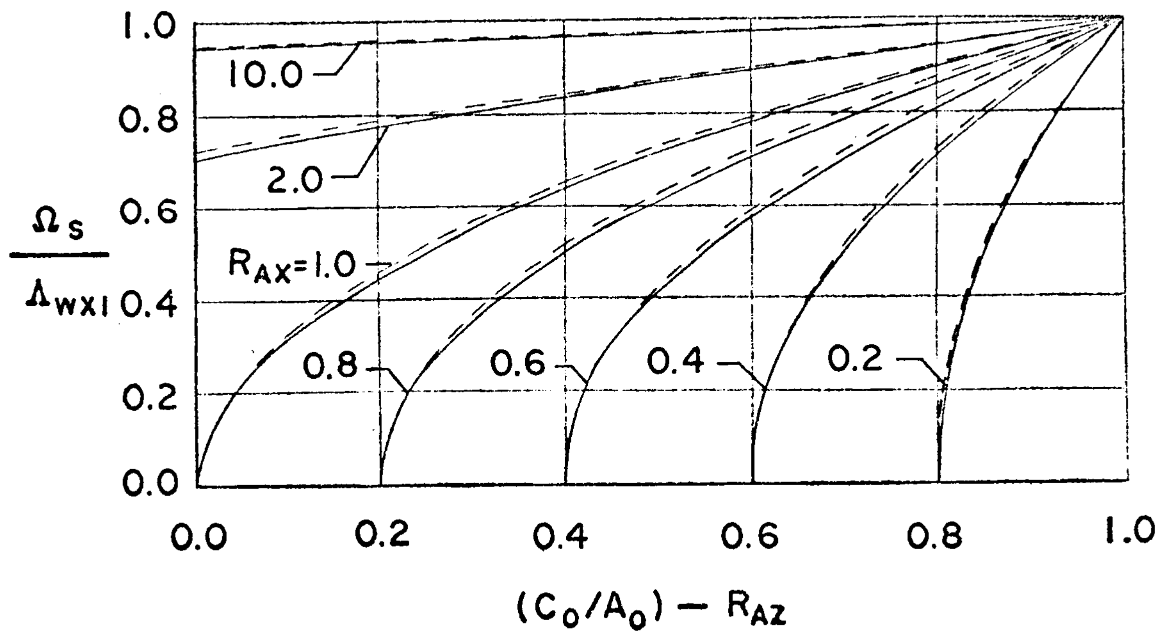


Figure 4

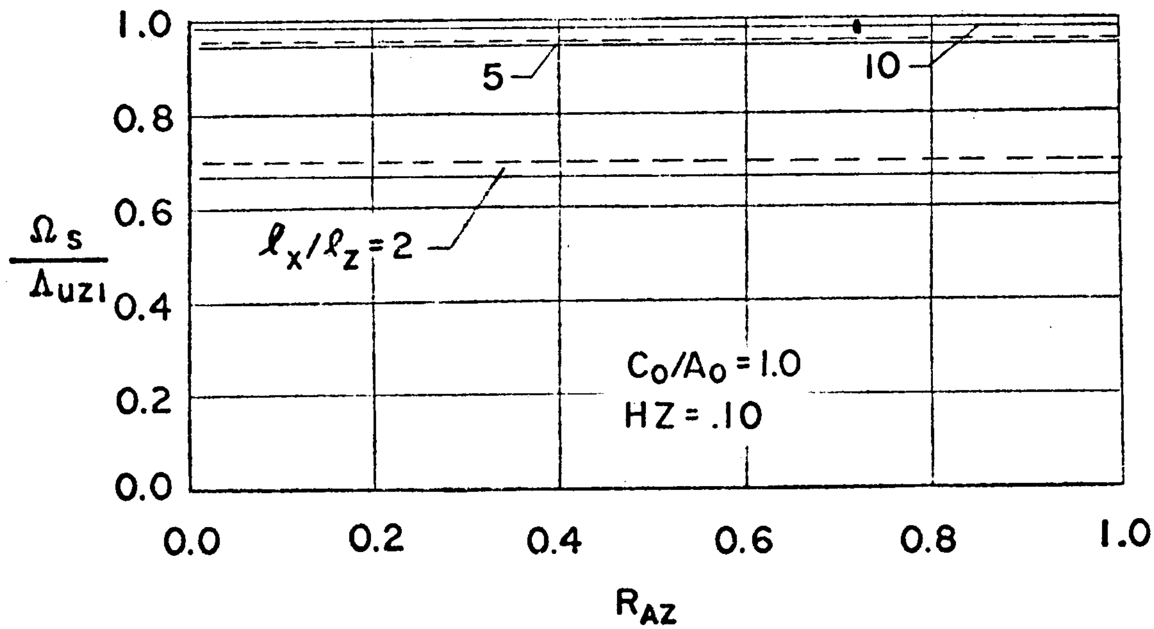
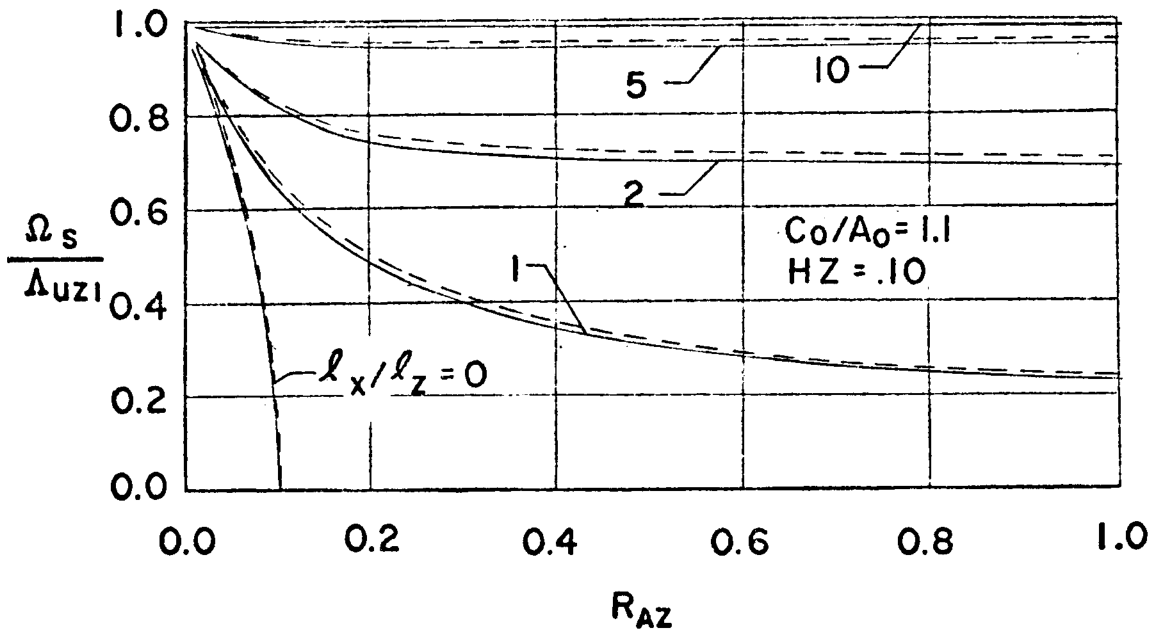


Figure 5



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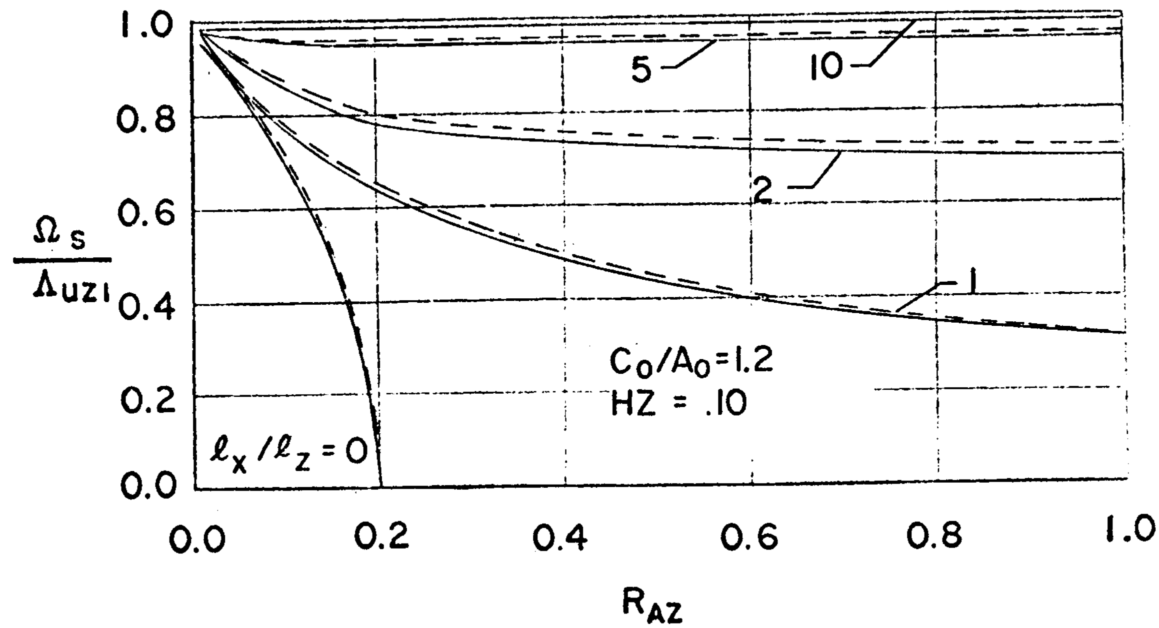
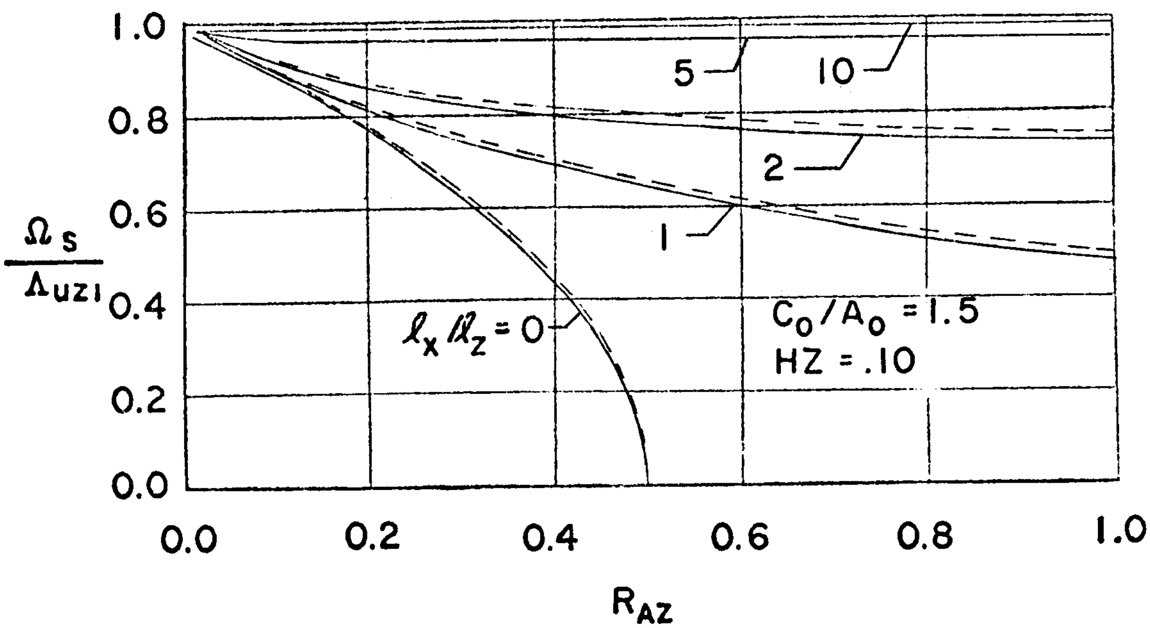


Figure 7



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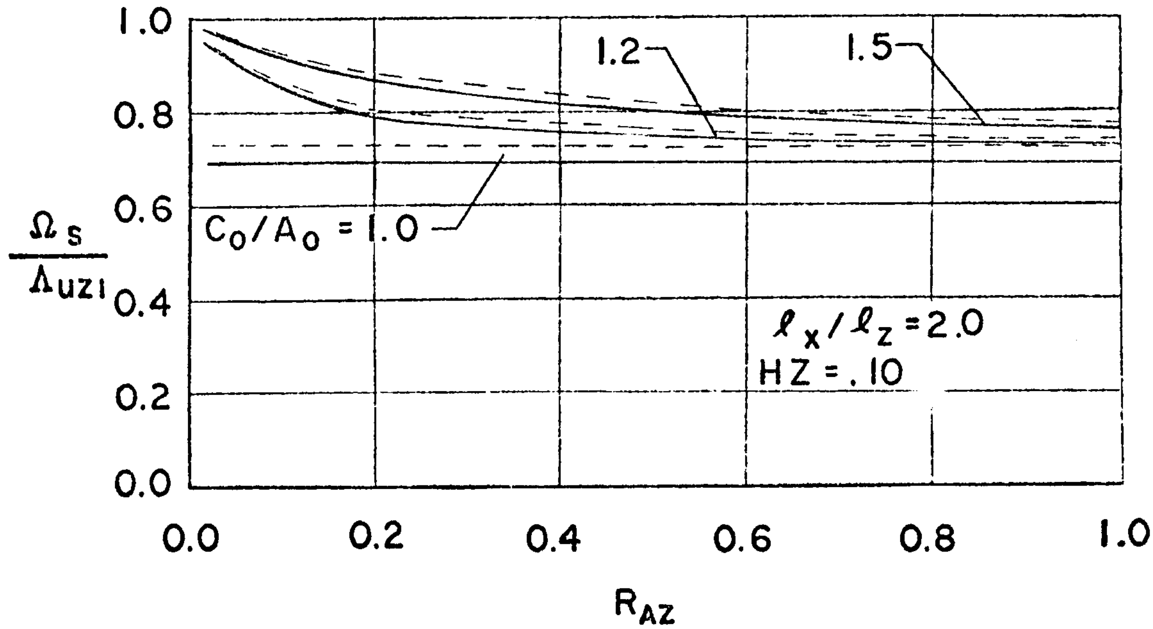

Figure 9

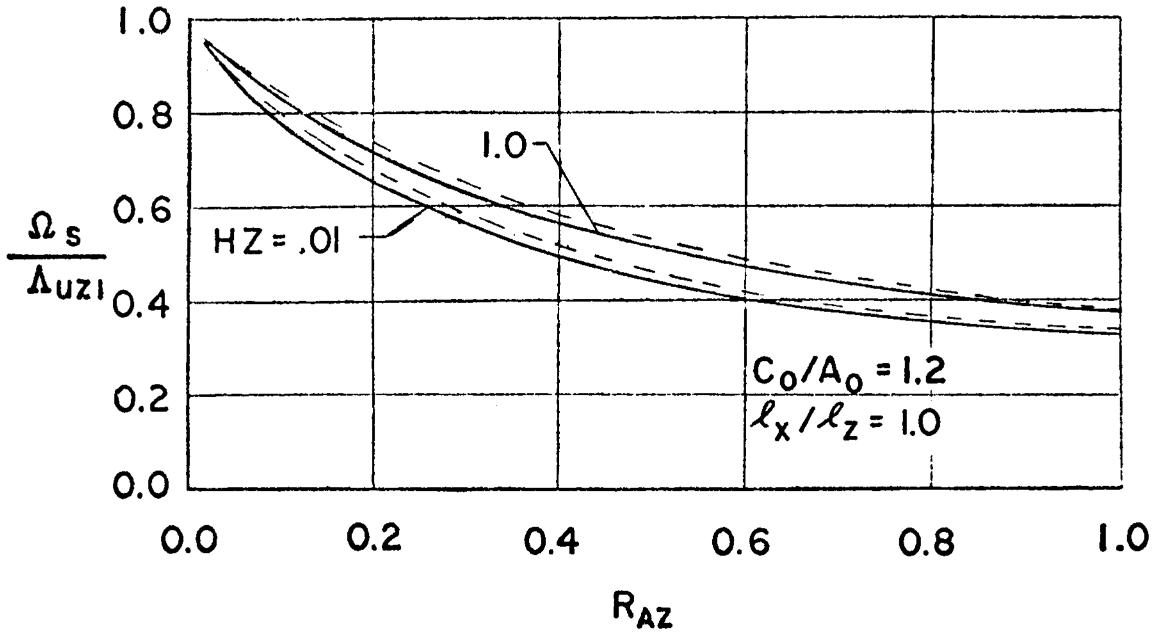

Figure 10

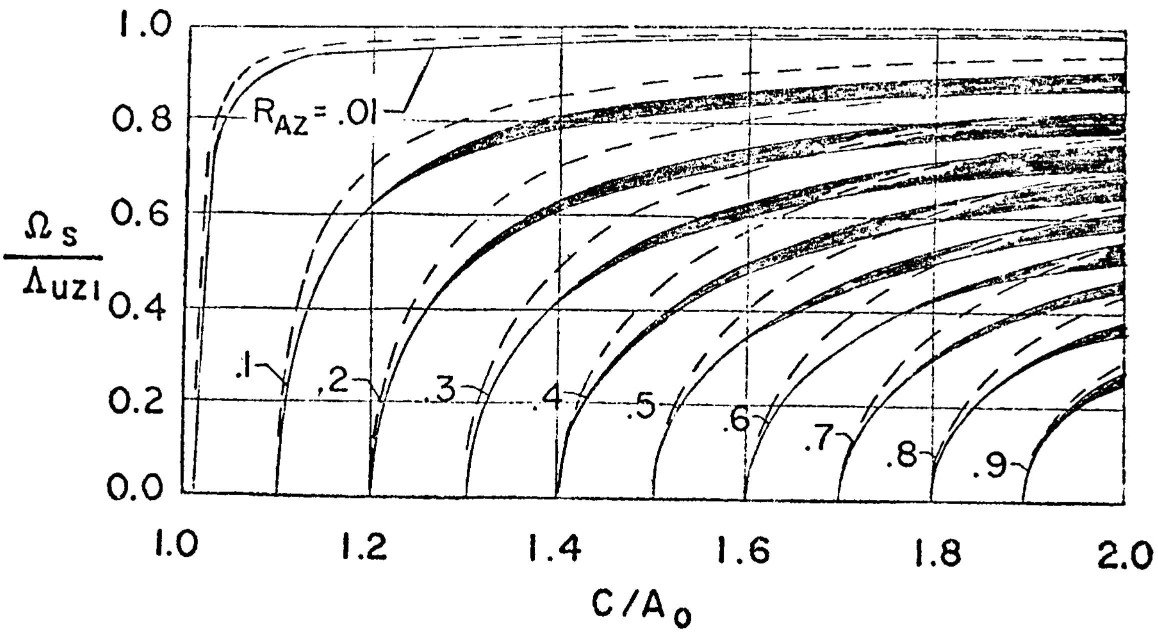

Figure 11

- - - - RESULTS OF PRESENT INVESTIGATION

——— RESULTS OF REFERENCE 16